Orbifold Hurwitz numbers, topological recursion and ELSV-type formulae

Danilo Lewanski, Amsterdam University

Combinatorics and interactions, CIRM, January 13th, 2017

Plan

§.1 ELSV-type formulae

§.2 Topological recursion

§.3 Givental theory

§.4 Identification between topological recursion and Givental theory

§.5 DOSS Identification for the spectral curve $\Sigma_{r,s}$ and consequences.

§.6 A new proof of JPT Formula.

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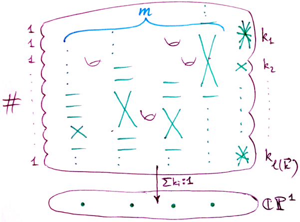
Simple Hurwitz numbers:

ELSV Formula [Ekedahl, Lando, Shapiro, Vainshtein]



$$= m! \left(\prod_i \frac{k_i^{k_i}}{k_i!} \right) \times \int_{\overline{\mathcal{M}}_{g,\ell(\vec{k})}} \frac{\sum_{i=0}^{g} (-1)^i \lambda_i}{\prod_i (1 - k_i \psi_i)}.$$

i) $h^{\bullet}_{g,\vec{k}} = \frac{1}{(\sum k_i)!} [C_{id}].C_{id} C^m_{(2)} C_{\vec{k}}$

ii) Riemann Hurwitz: $m = 2g - 2 + l(\vec{k}) + \sum k_i$.

iii) $\sum_{i=0}^{g} (-1)^i \lambda_i$ is the total Chern class of the dual of the Hodge bundle.

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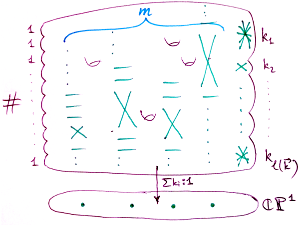

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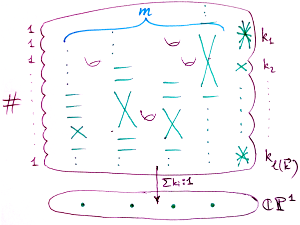

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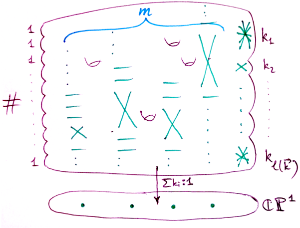

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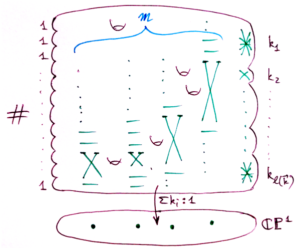
ELSV-type Formula [ALS; DK]



$$= \prod_i \binom{2k_i}{k_i} \int_{\overline{\mathcal{M}}_{g,\ell(\vec{k})}} \exp\Big(\sum_{l=1} A_l \kappa_l\Big) \times$$

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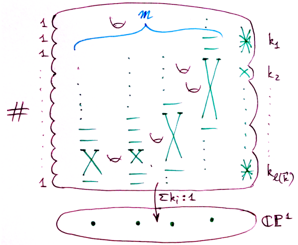

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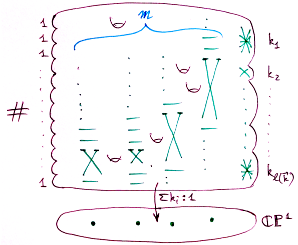

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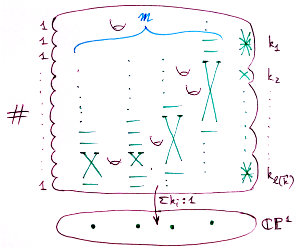

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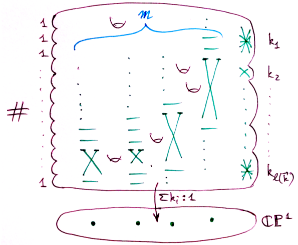

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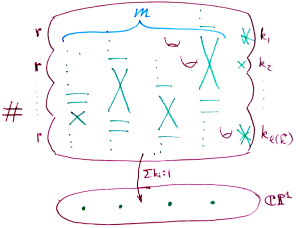
r-orbifold Hurwitz numbers:

JPT ELSV Formula [Johnson, Pandharipande, Tseng]
(Specialising $G = \mathbb{Z}/r\mathbb{Z}, U : 1 \mapsto e^{\frac{2\pi i}{r}}, \gamma = \emptyset$.)



$$= m! r^{2g-2+n-\sum\langle\mu_i\rangle} \prod \frac{k_i^{[k_i]}}{[k_i]!} \times$$

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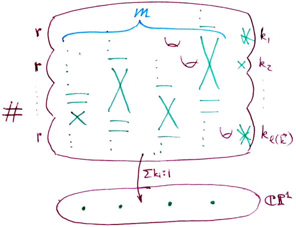

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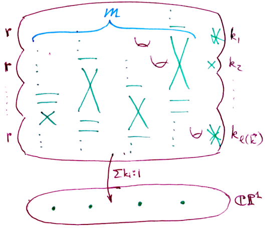

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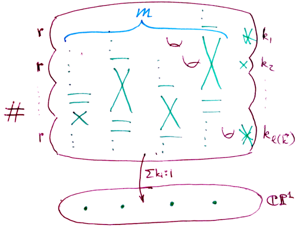

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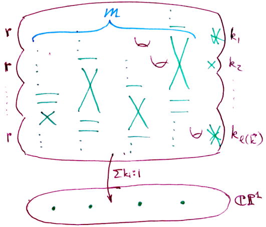
$\#$ 

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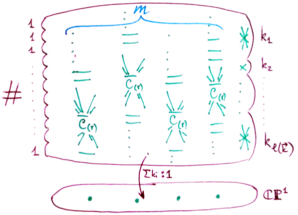
r-ELSV Formula [Zvonkine Conjecture]



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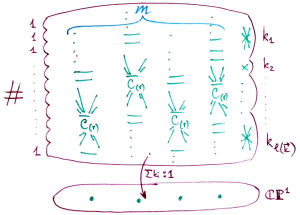

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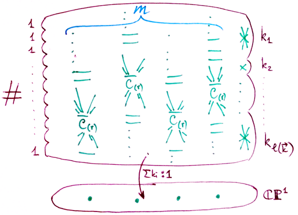

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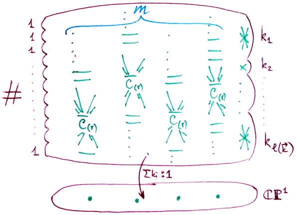

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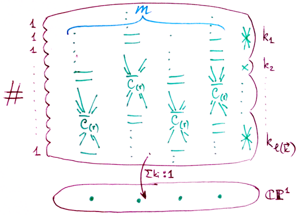

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ELSV-type formulae connect the intersection theory of a certain class on the moduli space of curves with the count of Hurwitz coverings with certain prescribed conditions.

Structure of ELSV-type formulae:

$$\text{Hurwitz numbers}^{\circ,condition}_{g,\vec{k}} = c \times NonPoly(k_i) \times Int(\overline{\mathcal{M}}_{g,\ell(\vec{k})}, Class, k_i)$$

Remark

These classes define a semi-simple cohomological field theory:

$$C_{g,n} : V^{\otimes n} \longrightarrow H^{even}(\overline{\mathcal{M}}_{g,n})$$
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ELSV-type formulae connect the intersection theory of a certain class on the moduli space of curves with the count of Hurwitz coverings with certain prescribed conditions.

Structure of ELSV-type formulae:

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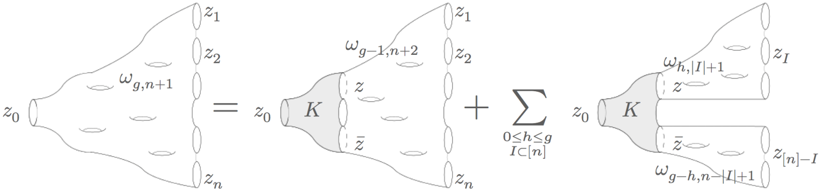
§.1 ELSV-type formulae: connect the intersection theory of a certain class on the moduli space of curves with the count of Hurwitz coverings with certain prescribed conditions.

§.2 Topological recursion

Topological recursion

Input: Spectral curve:= $(\Sigma, x(z), y(z), B(z_1, z_2))$

Output: $\omega_{g,n}$ symmetric differentials defined on on $\Sigma^{\times n}$ by a universal recursion on $2g-2+n$.



$$\text{Spectral Curve} \xrightarrow{\text{Topological recursion}} \text{Invariants } \omega_{g,n}.$$

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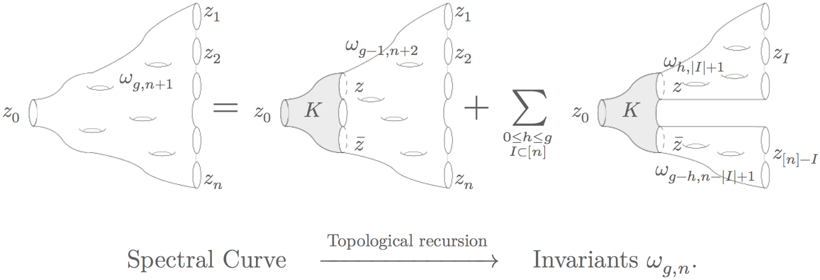
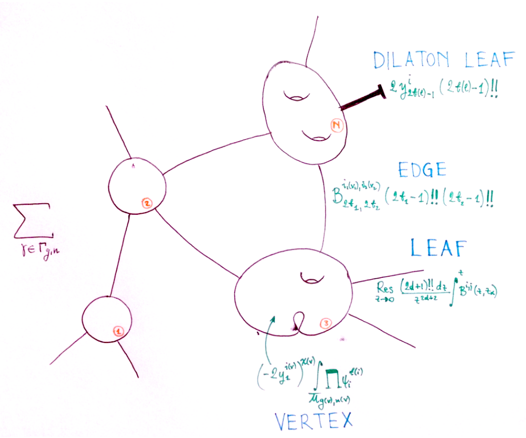


Correlation differentials $\omega_{g,n}$ as sum over stable graphs:



Correlation differentials $\omega^{\Sigma}_{g,n}$ and cohomological field theories:

Theorem (Eynard, DOSS)

$$\omega^{\Sigma}_{g,n} = \sum_{\substack{a_1,\dots,a_n\\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} C_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{z_j}\frac{\mathrm{d}}{\mathrm{d}z_j}\right)^{d_j} \xi_{i_j}(x_j)\right)$$

The non polynomial part comes from the expansion of ξ in x and its derivatives: the coefficients of the correlation differentials $\omega^{\Sigma}_{g,n}$ have the structure of right hand sides of ELSV-type formulae.

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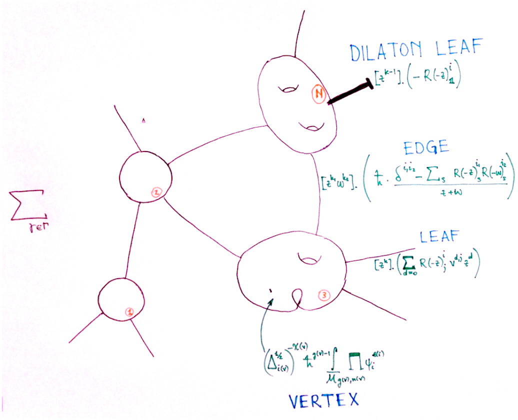
§.3 Givental theory

Givental theory

Theorem (Givental, Teleman)

Semisimple cohomological field theories are classified by Givental R matrices (matrix-valued power series): $C_{g,n} = (R^C.\alpha^{C,top})_{g,n}$



Equivalence between ELSV-type formulae and TR for certain Σ

TR: Spectral curve $\Sigma \longrightarrow$ Invariants

$$\omega^{\Sigma}_{g,n} = \mathrm{d}_1 \dots \mathrm{d}_n F_{g,n} = \mathrm{d}_1 \dots \mathrm{d}_n \sum_{\vec{k}:\ell(\vec{k})=n} h^{\circ,condition}_{g,\vec{k}} x_i^{k_i}$$

On the other side we know that:

$$\omega^{\Sigma}_{g,n} = \sum_{\substack{a_1,\dots,a_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} C_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{z_j} \frac{\mathrm{d}}{\mathrm{d}z_j} \right)^{d_j} \xi_{i_j}(x_j) \right)$$

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Computing the ingredients R, α, ξ (by equating monomials in x_i) means proving the equivalence of theorems:

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§.2 Topological recursion: the coefficients of the correlation differentials $\omega^{\Sigma}_{g,n}$ have the structure of right hand sides of ELSV-type formulae.

§.3 Givental theory: semisimple cohft are classified by the action of the Givental group

§.4 Identification between topological recursion and Givental theory

DOSS Identification

Question: How to compute the ingredients R, α, ξ directly from the spectral curve?

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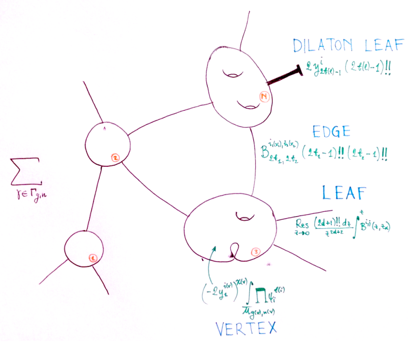
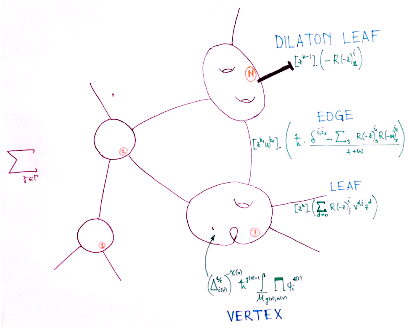
Answer: Compare each contribution on the stable graphs separately.





DOSS Identification: explicit formulae for $R^{\Sigma}, \alpha^{\Sigma}, \xi^{\Sigma}$.

i). The local coordinates w_i on U_i, $i = 1, \dots, r$, are chosen such that $w_i(p_i) = 0$ and $x = (c_i w_i)^2 + x_i$.

ii). Topological field theory: $\eta(e_i, e_j) = \delta_{ij}$, and

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Plan

§.1 ELSV-type formulae

§.2 Topological recursion

§.3 Givental theory

§.4 Identification between topological recursion and Givental theory

§.5 DOSS Identification for the spectral curve $\Sigma_{r,s}$ and consequences.

§.6 A new proof of JPT Formula.

DOSS identification for the spectral curve $\Sigma_{r,s}$:

$$\Sigma_{r,s} = \left(\mathbb{CP}^1, x(z) = -z^r + \log z, y(z) = z^s, B(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}\right)$$

i). Choose $c_i = 1/\sqrt{-2r}$ for $i = 1, \ldots, r$, $c = r^{1+s/r}/s$. Locally on U_i: $x = -\frac{w_i^2}{2r} + x(p_i)$.

ii). From idempotent to flat basis: $v_a := \sum_{i=0}^{r-1}(J^{ai}/r)e_i$. Topological field theory : $\eta(v_a, v_b) = r^{-1}\delta_{a+b \mod r}$, and $\alpha_{g,n}^{\Sigma_{r,s},top}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1}\delta_{a_1+\cdots+a_n-s(2g-2+n) \mod r}$.

iii). $R(\zeta) = \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{diag}_{a=1}^r B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)}\zeta^k\right)$

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Chiodo's classes:

For integers $r \geq 1$, $1 \leq a_1, \ldots, a_n \leq r, 0 \leq s \leq r$ such that

$$(2g-2+n)s - \sum a_i \in r\mathbb{Z}$$

$$\mathrm{Ch}_{g,n}(r,s;a_i) := \epsilon_* \exp\left(-\sum_{k=1}(k-1)!\mathrm{ch}_k(r,s;a_i)\right) \in H^*(\overline{\mathcal{M}}_{g,n})$$

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Theorem (LPSZ)

The correlation differentials $\omega_{g,n}^{\Sigma_{r,s}}$ obtained via topological recursion from the spectral curve $\Sigma_{r,s}$ take the following explicit form:

$$\omega_{g,n}^{\Sigma_{r,s}}(z_1,\ldots,z_n) = \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \frac{r^{2g-2+n+m}}{s^{2g-2+n}} \prod_{j=1}^{n} \frac{\left(\frac{k_j}{r}\right)^{[k_j]}}{[k_j]!} \times$$

$$\times \sum_{k_1,\ldots,k_n=1}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathrm{Ch}_{g,n}\left(r,s;r-\langle\vec{k}\rangle\right)}{\prod_{j=1}^{n}(1-\frac{k_i}{r}\psi_i)} e^{\sum k_j x_j}$$

where $m = \left((2g-2+n)s + \sum_{j=1}^{n} k_j\right)/r$.

Proposition

JPT's class $\pi_* \sum_{i\geq 0}^{g}(-1)^i\lambda^i = \mathrm{Ch}_{g,n}(r,r;r-\langle\vec{k}\rangle)$ Chiodo's class

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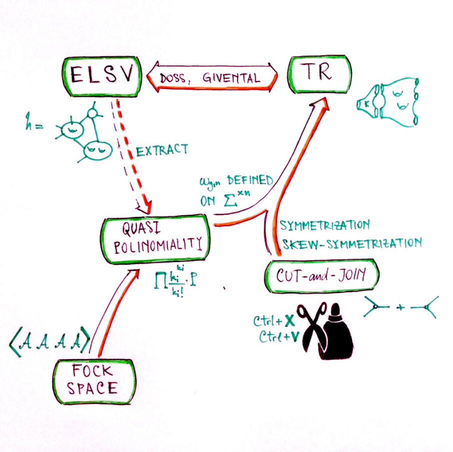
Corollaries:

Case	Topological recursion		ELSV-type formula
$s=1$ $r=1$	*The standard Hurwitz numbers* $h^{\circ}_{g,\vec\mu}$ *are generated by* $\left.\begin{aligned} -z+\log z &= x(z)\\ z &= y(z)\end{aligned}\right\}$	$\Longleftrightarrow$	*The ELSV formula* $\frac{h^{\circ}_{g;\vec\mu}}{b!} = \prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g^{\vee}(1)}{\prod_{j=1}^{n}(1-\mu_j\psi_j)},$ *holds, where:* $b=2g-2+n+\lvert\vec\mu\rvert.$
$s=r$	*The r-orbifold Hurwitz numbers* $h^{\circ,r}_{g,\vec\mu}$ *are generated by* $\left.\begin{aligned} -z^r+\log z &= x(z)\\ z^r &= y(z)\end{aligned}\right\}$	$\Longleftrightarrow$	*The Johnson-Pandharipande-Tseng formula* $\frac{h^{\circ,[r]}_{g;\vec\mu}}{b!} = r^b\prod_{i}\frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{p_*\sum_{i\geq 0}(-1)^i\lambda_i}{\prod_{j=1}^{n}(1-\frac{\mu_j}{r}\psi_j)},$ *holds, where:* $b=2g-2+n+\lvert\vec\mu\rvert/r.$
$s=1$	*The r-spin Hurwitz numbers* $h^{\circ,spin-r}_{g,\vec\mu}$ *are generated by* $\left.\begin{aligned} -z^r+\log z &= x(z)\\ z &= y(z)\end{aligned}\right\}$	$\Longleftrightarrow$	*The r-spin ELSV formula* $\frac{h^{\circ,r-spin}_{g;\vec\mu}}{b!} = r^{b-\chi}\prod_{i}\frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\mathrm{Ch}_{g,n}(r,1,r-\langle\vec\mu\rangle)}{\prod_{j=1}^{n}(1-\frac{\mu_j}{r}\psi_j)},$ *holds, where:* $b=(2g-2+n+\lvert\vec\mu\rvert)/r.$

Question: Can we get a new proof of JPT?



Conclusion: It suffices to prove quasi-polynomiality. Let us use $\mathcal{A}$ operators acting on the Fock space.

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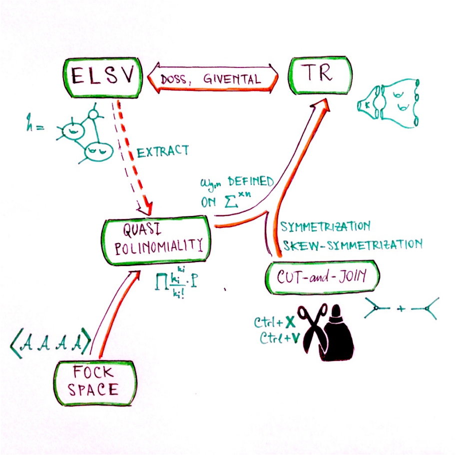

Conclusion: It suffices to prove quasi-polynomiality. Let us use $\mathcal{A}$ operators acting on the Fock space.

Semi-infinite wedge formalism

Vector space $V := \bigoplus_{c\in\mathbb{Z}} V_c$ spanned by the vectors that are obtained from

$$|0\rangle := z^{-1/2} \wedge z^{-3/2} \wedge z^{-5/2} \wedge \cdots$$

by applying a finite number of the operators

$$\psi_i := z^i \wedge \text{ and } \psi_i^* := \frac{\partial}{\partial z^i} \qquad i \in \mathbb{Z} + 1/2.$$

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$$|0\rangle \in V_0, \ \deg \psi_i = 1, \ \deg \psi_i^* = -1, \quad i \in \mathbb{Z} + 1/2.$$

Basis for V_0:

$$v_\lambda := z^{\lambda_1 - 1/2} \wedge z^{\lambda_2 - 3/2} \wedge z^{\lambda_3 - 5/2} \wedge \cdots$$

for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} \geq 0 \geq \cdots)$ Young diagram.

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Classical operators (Okounkov-Pandharipande)

Define the following operators on V_0:

$$E_{i,j} := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i & \text{if } j < 0 \end{cases}.$$

$$\mathcal{F}_n := \sum_{k \in \mathbb{Z}+1/2} \frac{k^n}{n!} E_{k,k}, \qquad \mathcal{F}_0 = C = \text{charge}, \qquad \mathcal{F}_1 = E = \text{energy}$$

$$\mathcal{E}_n(z) := \sum_{k \in \mathbb{Z}+1/2} e^{z(k-n/2)} E_{k-n,k} + \delta_{0,n}/\zeta(z)$$

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Semi-infinite wedge formalism for r-orbifold Hurwitz numbers

$$H^{\bullet,[r]}(\vec{k},u) := \sum_{g=0} h^{\bullet,[r]}_{g,\vec{k}} \frac{u^m}{m!},$$

$$= \sum_{g=0} \left\langle e^{\frac{\alpha_r}{r}} \mathcal{F}_2^m \prod_i \frac{\alpha_{-k_i}}{k_i} \right\rangle^{\bullet} \frac{u^m}{m!}$$

$$= \frac{1}{\prod_i k_i} \left\langle \prod_i e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_{-k_i} e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} \right\rangle^{\bullet}$$

$$= r^{\sum \langle k_i \rangle} u^{\frac{\sum k_i}{r}} \times \left(\prod \frac{k_i^{[k_i]}}{[k_i]!} \right) \times \frac{\left\langle \prod \mathcal{A}^{[r]}_{\langle k_i \rangle}(k_i,u) \right\rangle^{\bullet}}{k_1 \cdot \cdots \cdot k_n}$$

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\begin{aligned}
H^{\bullet,[r]}(\vec{k},u) :&= \sum_{g=0} h^{\bullet,[r]}_{g,\vec{k}} \frac{u^m}{m!}, \\
&= \sum_{g=0} \left\langle e^{\frac{\alpha_r}{r}} \mathcal{F}_2^m \prod_i \frac{\alpha_{-k_i}}{k_i} \right\rangle^{\bullet} \frac{u^m}{m!} \\
&= \frac{1}{\prod_i k_i} \left\langle \prod_i e^{\frac{\alpha_r}{r}} e^{u\mathcal{F}_2} \alpha_{-k_i} e^{-u\mathcal{F}_2} e^{-\frac{\alpha_r}{r}} \right\rangle^{\bullet} \\
&= r^{\sum \langle k_i \rangle} u^{\frac{\sum k_i}{r}} \times \left(\prod \frac{k_i^{[k_i]}}{[k_i]!} \right) \times \frac{\left\langle \prod \mathcal{A}^{[r]}_{\langle k_i \rangle}(k_i,u) \right\rangle^{\bullet}}{k_1 \cdot \dots \cdot k_n}
\end{aligned}
$$

$$
\mathcal{A}^{[r]}_{\eta}(z,u) = r^{-\eta/r} (\mathcal{S}(ruz))^{\frac{z-\eta}{r}} \sum_{k\in\mathbb{Z}} \frac{(\mathcal{S}(ruz))^k z^k}{(\frac{z-\eta}{r}+1)_k} \mathcal{E}_{kr-\eta}(uz) \quad \text{(Johnson)}
$$

Idea of the proof:

i). Enough to show polynomiality of connected correlators for k_1, by symmetry.

ii). Show rationality in k_1, for fixed $[k_2], \dots, [k_n], \langle k_1 \rangle, \dots, \langle k_n \rangle$ and fixed power of u, using vanishing near the covacuum and imposing zero total energy.

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v). This condition implies the vanishing of the residues in all connected correlators, obtain from the disconnected by inclusion-exclusion formula, except in the two cases $(g,n)=(0,1),(0,2)$ corresponding to the unstable correlators.

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Quasi-Polynomiality Results:

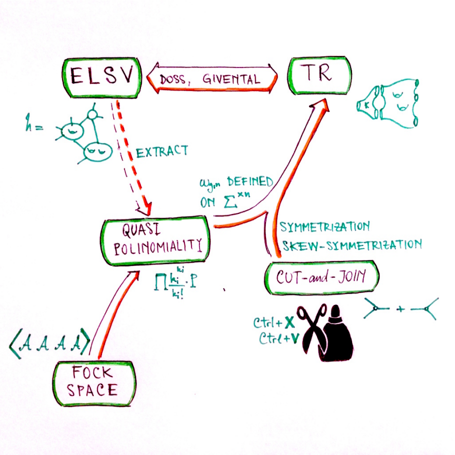
DLPS):
$$\frac{h^{\circ,[r]}_{g;\vec{k}}}{m!} = \prod_i \frac{k_i^{[k_i]}}{[k_i]!} \quad P^{\langle\vec{k}\rangle,[r]}_{g,n}(k_1,\dots,k_n)$$

KLS):
$$\frac{h^{\circ,<}_{g;\vec{k}}}{m!} = \prod_i \binom{k_i-1}{[k_i]} \quad P^{\langle\vec{k}\rangle,<}_{g,n}(k_1,\dots,k_n)$$

KLS):
$$\frac{h^{\circ,\leq}_{g;\vec{k}}}{m!} = \prod_i \binom{k_i+[k_i]}{k_i} \quad P^{\langle\vec{k}\rangle,\leq}_{g,n}(k_1,\dots,k_n)$$

KLPS):
$$\frac{h^{\circ,r-spin}_{g;\vec{k}}}{m!} = \prod_i \frac{k_i^{[k_i]}}{[k_i]!} \quad P^{\langle\vec{k}\rangle,r-spin}_{g,n}(k_1,\dots,k_n)$$

for stable (g,n), where P are polynomials in $k_1,\dots,k_n$ whose coefficients depend on the parameters $\langle k_i\rangle$.



Corollary

$$(\text{Def.}) \overset{DLPS}{\Longrightarrow} (\text{Quasi-Poly}) \oplus (\text{Cut-and-Join}) \underset{DLN}{\overset{BHSLM}{\Longrightarrow}} (TR) \overset{LPSZ}{\Longrightarrow} (JPT)$$

Rèsumè

§.1 ELSV-type formulae: connect the intersection theory of a certain class on the moduli space of curves with the count of Hurwitz coverings with certain prescribed conditions.

§.2 Topological recursion: the coefficients of the correlation differentials $\omega^{\Sigma}_{g,n}$ have the structure of right hand sides of ELSV-type formulae.

§.3 Givental theory: semisimple cohft are classified by the action of the Givental group

§.4 Identification between topological recursion and Givental theory: computing the ingredients R, α, ξ means proving the equivalence of theorems:

$$TR(h^{\circ,condition}_{g,\vec{k}}) \iff ELSV(h^{\circ,condition}_{g,\vec{k}})$$

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The case of $\Sigma_{r,s}$

- Consider the particular spectral curve $\Sigma_{r,s} = \left(\mathbb{CP}^1, x(z) = -z^r + \log z, y(z) = z^s, B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}\right)$ and compute the ingredients $R^{\Sigma_{r,s}}, \alpha^{\Sigma_{r,s}}, \xi^{\Sigma_{r,s}}$ by DOSS.
- Obtain the equivalence of theorems

$$TR(h^{\circ,r,s}_{g,\vec{k}}) \iff ELSV(h^{\circ,r,s}_{g,\vec{k}})$$

and recover known equivalences for $s = 1$ and for $s = r = 1$.
- For the case $s = r$ obtain $TR(h^{\circ,r-orbifold}_{g,\vec{k}}) \iff JPT$
- As a consequence, JPT implies a proof of TR. However, proofs of TR for orbifold Hurwitz are already obtained from JPT by extracting the quasi-polynomiality property and combining it with cut-and-join equation, but this implies forgetting a lot of informations from JPT.
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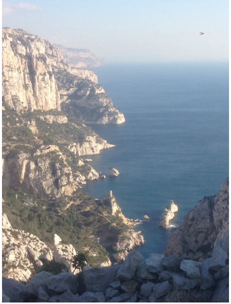
Merci pour votre attention



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