Nonmixing sets of algebraic $\mathbb{Z}^d$-actions

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Mixing

Let $T\colon \mathbf{n} \mapsto T^{\mathbf{n}}$ be a $\mathbb{Z}^d$-action by measure-preserving transformations of a probability space $(X, \mathcal{S}, \mu)$. The action T is *mixing* if

$$\lim_{\mathbf{n}\to\infty} \mu(B_1 \cap T^{-\mathbf{n}} B_2) = \mu(B_1)\mu(B_2)$$

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More generally, the action T is r-mixing with $r \geq 2$ if, for all $B_1, \dots, B_r \in \mathcal{S}$,

$$\mu\Bigl(\bigcap_{i=1}^{r} T^{-\mathbf{n}_i} B_i\Bigr) \longrightarrow \prod_{i=1}^{r} \mu(B_i) \text{ as } |\mathbf{n}_i - \mathbf{n}_j| \to \infty \text{ for } 1 \leq i < j \leq r.$$

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Astrology is based on a breakdown of r-mixing for some appropriate $r \geq 3$.

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In 1978, Ledrappier gave a simple example of a mixing $\mathbb{Z}^2$-action which fails to be r-mixing for every $r \geq 3$.

Ledrappier's Example

Let σ be the shift-action $(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$ of $\mathbb{Z}^2$ on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$, and let $X_L \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ be the closed, shift-invariant subset (in fact, subgroup)

$$X_L = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x + \sigma^{(1,0)} x + \sigma^{(0,1)} x = 0\}.$$

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For every $x \in X_L$, $(n_1, n_2) \in \mathbb{Z}^2$ and $k \geq 0$,

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$$B \cap \sigma^{(-2^k,0)}(B) \cap \sigma^{(0,-2^k)}(B) = B \cap \sigma^{(-2^k,0)}(B).$$

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In fact, there exists a *nonmixing set* of size 3 in $\mathbb{Z}^2$: if $F = \{(0,0),(1,0),(0,1)\}$, then there exist sets $B_{\mathbf{n}} \,(= B) \in \mathcal{S} = \mathcal{B}_{X_L}$, $\mathbf{n} \in F$, such that

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If a $\mathbb{Z}^d$-action T has a nonmixing set of size r then it is obviously not r-mixing. What about the converse?

Higher order mixing for algebraic $\mathbb{Z}^d$-actions

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- If X not connected, α is mixing of every order if and only if it has completely positive entropy or, equivalently, the Bernoulli property (Lind-S-Ward, 1990, S-Ward, 1993, and Rudolph-S, 1995).

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- If α is expansive, then both the order of mixing and the collection of minimal nonmixing sets can be determined effectively (Derksen-Masser, 2012-2016).

Back to Ledrappier's Example

Let $d \geq 2$, and let $R_d^{(p)} = \mathbb{F}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in d variables with coefficients in the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, \dots, p-1\}$. Then $R_d^{(p)} \cong \widehat{\sum_{\mathbb{Z}^d} \mathbb{F}_p}$:
for $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \in R_d^{(p)}$ and $x = (x_{\mathbf{n}}) \in \mathbb{F}_p^{\mathbb{Z}^d}$,

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The shifts $\sigma^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, are automorphisms of the compact abelian group $\mathbb{F}_p^{\mathbb{Z}^d}$ dual to multiplication by $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ on $R_d^{(p)}$.

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Recall that Ledrappier's example is defined by

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The annihilator $X_L^{\perp}$ of the closed, shift-invariant subgroup $X_L \subset \mathbb{F}_2^{\mathbb{Z}^2}$ is the subgroup of $R_2^{(2)}$ consisting of all $f = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} u^{\mathbf{n}} \in R_2^{(2)}$ with

$$\langle f, x \rangle = 1 \iff \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} x_{\mathbf{n}} = 0 \text{ for every } x = (x_{\mathbf{n}}) \in X_L.$$

The dual of Ledrappier's Example

Since X_L is shift-invariant, $X_L^\perp$ is invariant under multiplication by $u^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2$, hence an ideal, and $g = 1 + u_1 + u_2 \in X_L^\perp$ by definition of X_L. It follows that $X_L^\perp$ is the principal ideal $\mathfrak{p} = (g) = g \cdot R_2^{(2)} \subset R_2^{(2)}$, and that

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If $\mathbb{k} = \mathrm{Frac}(R_2^{(2)}/\mathfrak{p}) \supset R_2^{(2)}/\mathfrak{p}$ is the field of fractions of the domain $R_2^{(2)}/\mathfrak{p}$, then $g = g^{2^l} = 0$ in $\mathbb{k}$ for every $l \geq 1$, so that we get an infinite sequence of equations in $\mathbb{k}$ of the form

$$\sum\nolimits_{\mathbf{n} \in F} u^{k_i \mathbf{n}} \cdot a_{\mathbf{n}} = 0, \quad i \geq 1,$$

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Every $f = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{p}$ leads to a similar sequence of equations in $\Bbbk$, where $F' = \mathrm{supp}(f) = \{\mathbf{n} \in \mathbb{Z}^2 : f_{\mathbf{n}} \neq 0\}$. Hence the support of every $f \in \mathfrak{p}$ is a nonmixing set for Ledrappier's example.

Additive relations in dual modules

Ledrappier's example illustrates a general fact: if α is a $\mathbb{Z}^d$-action by automorphisms of a compact abelian group X, then the dual group $M = \widehat{X}$ is a module over the ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ with module operation

$$h \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} \hat{\alpha}^{\mathbf{n}}(a)$$

for every $a \in \hat{X}$ and $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in R_d$, where $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ for all $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$.

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For obvious reasons $M = \widehat{X}$ is called the dual module of the $\mathbb{Z}^d$-action α; conversely, every module M over R_d defines a dual $\mathbb{Z}^d$-action $\alpha = \alpha_M$ by automorphisms of a compact abelian group $X = \widehat{M}$.

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For obvious reasons $M = \widehat{X}$ is called the dual module of the $\mathbb{Z}^d$-action α; conversely, every module M over R_d defines a dual $\mathbb{Z}^d$-action $\alpha = \alpha_M$ by automorphisms of a compact abelian group $X = \widehat{M}$.

By using Fourier expansion one sees that α_M is not r-mixing if and only if there exist elements $a_1, \dots, a_r$ in M, not all equal to zero, with

$$u^{\mathbf{n}_k^{(1)}} \cdot a_1 + \cdots + u^{\mathbf{n}_k^{(r)}} \cdot a_r = 0 \qquad (1)$$

for some sequence $((\mathbf{n}_k^{(1)}, \dots, \mathbf{n}_k^{(r)}), k \geq 1)$ in $(\mathbb{Z}^d)^r$ with $\mathbf{n}_k^{(i)} - \mathbf{n}_k^{(j)} \to \infty$ for $i \neq j$.

Additive relations in fields

In exactly the same way one sees that α_M has a nonmixing set $F \subset \mathbb{Z}^d$ if and only if if there exist elements $a_{\mathbf{n}}$, $\mathbf{n} \in F$, in M, not all equal to zero, with

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By using prime filtrations we can replace the module M in (1) or (2) by the module $N = R_d/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R_d$ associated with M and consider such equations in the field of fractions $\mathrm{Frak}(R_d/\mathfrak{p})$ of $R_d/\mathfrak{p}$.

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If the group X carrying the action α is connected, the characteristic of $\mathrm{Frak}(R_d/\mathfrak{p})$ will be zero for every associated prime ideal $\mathfrak{p}$ of M. If not, $\mathrm{char}(\mathrm{Frak}(R_d/\mathfrak{p}))$ will be positive for some associated prime $\mathfrak{p}$ of M.

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In the latter case there exist a rational prime $p \geq 2$ and a prime ideal $\mathfrak{q} \subset R_d^{(p)}$ such that $N = R_d/\mathfrak{p} \cong R_d^{(p)}/\mathfrak{q}$.

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For simplicity we will call actions of the form $\alpha_{R_d^{(p)}/\mathfrak{q}}$ actions of Ledrappier type.

A theorem by Mahler and its consequences

Theorem (Mahler, 1935). Let $\Bbbk$ be a field of characteristic 0, $r \geq 2$, and let $c_1, \dots, c_r$ be nonzero elements of $\Bbbk$. If we can find nonzero elements $x_1, \dots, x_r$ in $\Bbbk$ such that the equation

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

holds for infinitely many $k \geq 0$, then there exist integers $s \geq 1$ and i, j with $1 \leq i < j \leq r$ such that $x_i^s = x_j^s$.

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Corollary (S, 1989). Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then every nonempty finite subset $S \subset \mathbb{Z}^d$ is mixing.

An S-unit theorem and its consequences

Theorem (Schlickewei, 1990; van der Poorten-Schlickewei, 1991; Evertse-Schlickewei-Schmidt, 2002). Let $\mathbb{k}$ be a field of characteristic 0 and G a finitely generated multiplicative subgroup of $\mathbb{k}^\times = \mathbb{k} \smallsetminus \{0\}$. If $r \geq 2$ and $(c_1, \dots, c_r) \in (\mathbb{k}^\times)^r$, then the equation

$$\sum_{i=1}^{r} c_i x_i = 1 \qquad (3)$$

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Corollary (S-Ward, 1993). Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then α is mixing of every order.

Mahler's theorem has the following analogue in positive characteristic.

Theorem (Masser, 1985; Kitchens-S, 1993). Let $\mathbb{k}$ be a field of characteristic $p \geq 2$, $r \geq 2$, and let $(x_1, \dots, x_r) \in (\mathbb{k}^\times)^r$. The following conditions are equivalent:

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- There exists a rational number $s > 0$ such that the subset $\{x_1^s, \ldots, x_r^s\}$ of the algebraic closure $\overline{\Bbbk}$ of $\Bbbk$ is linearly dependent over the algebraic closure $\overline{\mathbb{F}}_p \subset \overline{\Bbbk}$ of $\mathbb{F}_p$.

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The following example illustrates the consequences of this result for Ledrappier-like systems.

An example

Let $g = \sum_{\mathbf{n} \in \mathbb{Z}^d} g_{\mathbf{n}} u^{\mathbf{n}} \in R_2^{(2)}$. Then the R_2-module $M = R_2^{(2)}/(g)$ is dual to the closed, shift-invariant subgroup

$$X_M = \Big\{ (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} : \sum_{\mathbf{n} \in \mathbb{Z}^d} g_{\mathbf{n}} x_{\mathbf{m}+\mathbf{n}} = 0 \text{ for every } \mathbf{m} \in \mathbb{Z}^2 \Big\} \subset \mathbb{F}_2^{\mathbb{Z}^2}.$$

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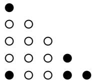
Clearly, $F = \mathrm{supp}(g)$ is a nonmixing set, so that α_M is not mixing of order $|F|$. However, Masser's result may yield smaller nonmixing sets – and hence a lower order of mixing: if

$$g = 1 + u_1^3 + u_1^4 + u_1^3 u_2 + u_2^4,$$

then

$F = \mathrm{supp}(g) =$ 

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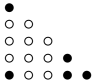
$F = \operatorname{supp}(g) =$ 

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In fact, there are 54 irreducible polynomials g of degree 4 in $R_2^{(2)}$ for which $F' =$  is a minimal nonmixing set (S, 1995).

The order of mixing

If a Ledrappier-like system has a nonmixing set of size r, then it can obviously not be r-mixing. The reverse implication was explored in examples (e.g., Einsiedler-Ward, 2003), but the complete solution of this problem was again due to David Masser.

Theorem (Masser, 2004): Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. If every subset $S \subset \mathbb{Z}^d$ of cardinality $r \geq 2$ is mixing, then α is r-mixing.

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The proof of this result requires an analogue for positive characteristic of the S-unit theorem mentioned earlier. In positive characteristic, the condition of 'nonvanishing sub-sums' in the S-unit theorem guaranteeing finiteness of the set of solutions $(x_1, \dots, x_r)$ of an equation of the form

$$c_1 x_1 + \cdots + c_r x_1 = 1$$

is clearly insufficient: the equation

$$x_1 + x_2 = 1, \quad x_i \in \mathbb{F}_p(t),$$

has the solution $x_1 = t$, $x_2 = 1 - t$ but also, by Frobenius, $x_1 = t^{p^n}$, $x_2 = (1 - t)^{p^n}$ for every $n \geq 1$.

A diophantine reformulation of Masser's Theorem

Definition: Let $\Bbbk$ be a field of positive characteristic, $G \subset \Bbbk^\times$ a finitely generated group, and $r \geq 2$. An infinite subset $\Sigma \subset G^r$ is broad if, for each $g \in G$ and $1 \leq i < j \leq r$, there are only finitely many $(x_1, \ldots, x_r) \in \Sigma$ with $x_i/x_j = g$.

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As mentioned earlier, it suffices to prove Masser's Theorem for Ledrappier-like systems. Assume therefore that $\alpha = \alpha_{R_d/\mathfrak{p}}$ is Ledrappier-like and let $\Bbbk = \mathrm{Frak}(R_d/\mathfrak{p})$. Then Masser's Theorem becomes equivalent to the following statement.

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Theorem (Masser, 2004): Suppose that there exist $c_1, \ldots, c_r$ in $\mathbb{k}$ such that the equation $c_1x_1 + \cdots + c_rx_r = 1$ has a broad set of solutions $(x_1, \ldots, x_r)$ in G^r. Then there are $a_1, \ldots, a_r$ in $\mathbb{k}$ and $g_1, \ldots, g_r$ in G with the following properties.

- $g_i/g_j \neq 1$ for $1 \leq i < j \leq r$,
- $a_1g_1^k + \cdots + a_rg_r^k = 1$ for infinitely many $k \geq 1$.

Minimal nonmixing sets

Masser's Theorem implies that the order of mixing of an algebraic $\mathbb{Z}^d$-action α is completely determined by its smallest nonmixing sets (where 'smallest' refers to cardinality).

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Two nonmixing sets $S, S' \subset \mathbb{Z}^d$ for an algebraic $\mathbb{Z}^d$-action α will be called equivalent if $qS' = S - \mathbf{n}$ for some positive rational q and some $\mathbf{n} \in \mathbb{Z}^d$. The resulting equivalence class of each nonmixing set contains a unique 'smallest' representative $S' \subset \mathbb{Z}_+^d$.

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Theorem (Derksen-Masser, 2016): Let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be a mixing Ledrappier-type $\mathbb{Z}^d$-action. Then

$$r(\alpha) = \min\{|S| : S \subset \mathbb{Z}^d \text{ is } \alpha\text{-nonmixing}\}$$

can be effectively determined, and there exist only finitely many distinct equivalence classes of nonmixing sets of size r which can again be determined effectively.

Example (Derksen-Masser, 2016)

Let $\mathfrak{p} = (2, f)$ with $f = 1 + u_1 + u_1^3 + u_1^5 + u_1^6 + u_2 \in R_2$. Then $g = f \pmod 2$ is irreducible in $R_2^{(2)}$, and $S = \mathrm{supp}(g)$ is nonmixing of size 6. However, g is divisible by $h = 1 + u_1 + u_1^2 + u_2^{1/3}$ in $\mathbb{F}_2[u_1, u_2^{1/3}]$. By thinking of g as an element of $\mathbb{F}_2[u_1^{1/3}, u_2^{1/3}]$ and scaling things up we see that $S' = \{(0,0),(3,0),(6,0),(0,1)\}$ is nonmixing of size 4 (S, 1995). However, there are 4 other equivalence classes of nonmixing sets of size 4:

$$\{(0,0),(9,0),(6,1),(0,2)\}, \ \ \{(0,0),(9,0),(0,1),(3,1)\},$$
$$\{(3,0),(12,0),(0,1),(0,2)\}, \ \ \{(0,0),(18,0),(3,2),(0,3)\}.$$

Since there are no nonmixing sets of size 3, $\alpha_{R_2/\mathfrak{p}}$ is 3-mixing.

Example (Derksen-Masser, 2016)

Let $\mathfrak{p} = (2, 1 + u_1 + u_1^2 + u_2, 1 + u_1 + u_1^3 + u_3) \subset R_2$. Clearly, $\alpha = \alpha_{R_3/\mathfrak{p}}$ is not 4-mixing, since there are nonmixing sets of size 4, but this time there are exactly 134 distinct classes of minimal nonmixing sets of size 4. Two of them come from the generators of the ideal $\mathfrak{p}$:

$$\{(0,0,0),(1,0,0),(2,0,0),(0,1,0)\},$$
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This example is again 3-mixing.