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INTEGRAL POINTS ON MARKOFF TYPE CUBIC SURFACES AND DYNAMICS

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LUMINY, DEC 2016

JOINT WORK WITH

A. GHOSH &

and

J. BOURGAIN / A. GAMBURD.

$D \in \mathbb{Z}[x_1, \dots, x_n]$ OF DEGREE n.

[SO THAT FOR $|x| \leq T$ GENERIC NUMBERS IN $\mathbb{Z}$ (SIZE T^n) ARE ASSUMED $O(1)$ TIMES]

V_k : $D(x) = k$, AFFINE HYPERSURFACE

· INTERESTED IN $V_k(\mathbb{Z})$

$V_k(\mathbb{Z}) \neq \phi$; HASSE PRINCIPLE

$V_k(\mathbb{Z})$ INFINITE OR EVEN ZARISKI DENSE

$V_k(\mathbb{Z})$ SATISFIES STRONG APPROXIMATION.

$n=2$: BINARY QUADRATIC FORMS:

· OVER $\mathbb{Q}$: WELL BEHAVED IN TERMS OF LOCAL TO GLOBAL PRINCIPLES.

· OVER $\mathbb{Z}$: GAUSS' THEORY

CLASS- NUMBERS INTERVENE!

$n > 2$:

- IF V_k IS HOMOGENEOUS FOR AN AFFINE LINEAR GROUP ACTION ONE CAN SAY QUITE A LOT.

EXAMPLES:

(1) TORUS: $D(x) = \prod_{j=1}^{m} L_j(x)$

$L_j(x)$ LINEAR FORMS, I.E. D IS A NORM FORM. DIRICHLET'S UNIT THEOREM AND THE THEORY OF NORMS OF ELEMENTS IN ORDERS ALLOW FOR A STUDY (AGAIN CLASS NUMBERS INTERVENE).

(2) DISCRIMINANTS OF CUBICS:

$$D(x_1,x_2,x_3,x_4) = 18x_1x_2x_3x_4 + x_1^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2$$

THE DEGREE 4 IN 4 VARIABLES DISCR. OF BINARY CUBICS.

$GL_2(\mathbb{Z})$ ACTS LINEARLY ON $X_k(\mathbb{Z})$ WITH FINITELY MANY $h(k)$ ORBITS ($h(k)$ IS HARD TO STUDY)

$h(k)$ IS TYPICALLY SMALL

$$\sum_{k \le K}^{*} h(k) \sim \frac{\pi^2}{108} K \quad (\text{DAVENPORT})$$

AND IT IS EXPECTED THAT A POSITIVE PROPORTION OF k'S HAVE $h(k) \neq 0$ (IE $V_k(\mathbb{Z}) \neq \phi$), COHEN-LENSTA HEURISTICS.

FOR A GENERAL D LITTLE IS KNOWN ABOUT $X_k(\mathbb{Z})$ OR ANY OF THESE PROBLEMS.

WE DISCUSS SOME AFFINE VARIETIES THAT ARE DEFINED OVER $\mathbb{Z}$ AND FOR WHICH THE IS A <u>NON-LINEAR</u> DESCENT GROUP OF MORHISMS ACTING ON V.

THESE ARISE AS CHARACTER VARIETIES FOR REPRSENTATIONS OF

$$\pi_1(\Sigma_{g,n}) \longrightarrow SL_2$$

WE RESTRICT TO THE CASE OF CUBIC SURFACES.

INTEGRAL POINTS ON AFFINE CUBIC SURFACES:

TORI: ARE WELL UNDERSTOOD THANKS TO DIRICHLET'S UNIT THEOREM.

SPLIT CASE: $X_k : x_1 x_2 x_3 = k$

$h(k) = d_3(k)$, # OF DIVISORS THREE FACTORS

ANISOTROPIC CASE: $X_k : N(x_1, x_2, x_3) = k$

N – NORM FORM OF CUBIC ORDER $/\mathbb{Z}$

CONTROLLED BY THE UNIT GROUP $N(x) = 1$.

NOTE: $X_k(\mathbb{Z})$ NEVER SATISFIES STRONG APPROXIMATION.

$$X_k : \quad x_1^3 + x_2^3 + x_3^3 = k$$

VERY LITTLE IS KNOWN ABOUT $X_k(\mathbb{Z})$.

- LOCAL CONGRUENCE OBSTRUCTION $k \not\equiv 4, 5\ (9)$.
- HASSE PRINCIPLE: IF $k \not\equiv 4, 5\ (9)$ IS $X_k(\mathbb{Z}) \neq \phi$? IS $|X_k(\mathbb{Z})| = \infty$?
- THE STRONGEST FORM OF STRONG APPROXIMATION FAILS (CASSELS, HEATH-BROWN, COLLIOTE-THELENE/WITTENBERG)

MARKOFF LIKE SURFACES

MAIN EXAMPLE:

$$X_k : \quad x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = k$$

A key feature is that X_k has a large group of morphisims coming from

Vieta Transformations:

Fix x_2, x_3 then x_1, x_1' the two solutions to the quadratic equation

R_1 switches the roots

$$R_1((x_1, x_2, x_3)) = (x_2 x_3 - x_1, x_2, x_3)$$

similarly with R_2, R_3. $\quad R_j^2 = I$.

Γ is the group of polynomial morphisms of A^3 generated by R_1, R_2, R_3, permutations of the coords and switching the signs of two coordinates.

- Γ preserves $X_k(\mathbb{Z})$.

This allows for a descent procedure to examine $X_k(\mathbb{Z})$.

SUCH SURFACES AND AUTOMORHISMS ARISE IN VARIOUS CONTEXTS:

- X_k IS THE CHARACTER VARIETY OF REPRESENTATIONS OF $\pi_1(\Sigma_{1,1})$ INTO SL_2. HERE $\Sigma_{g,n}$ IS A n-PUNCTURED CURVE OF GENUS g. Γ IS THE MAPPING CLASS GROUP WITH ITS ACTION ON X_k.
- Γ AND X_k ARISE IN THE STUDY OF THE NONLINEAR MONODROMY GROUP OF THE PAINLEVE VI EQUATION.

DYNAMICS OF Γ ON $X_k(\mathbb{R})$ (W. GOLDMAN).

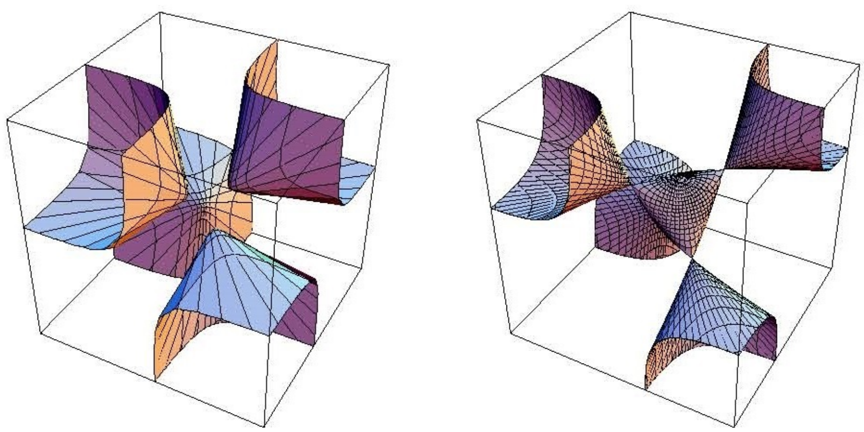
- FOR $k<0$, Γ ACTS PROPERLY ON $X_k(\mathbb{R})$
- FOR $0 \leq k < 4$, Γ ACTS ERGODICALLY ON THE COMPACT COMPONENTS AND PROPERLY ON THE OTHERS
- $k=4$, CAYLEY CUBIC ACTION IS LINEAR IN SUITABLE COORDS.
- $4<k\leq 20$, Γ ACTS ERGODICALLY ON $X_k(\mathbb{R})$.
- $k>20$ THERE IS AN OPEN WANDERING DOMAIN IN $X_k(\mathbb{R})$.

In all cases we have the group $\Gamma = \Gamma_S$ of affine polynomial morphisms generated by the Vieta transformations, acting on S and $S(\mathbb{Z})$, $(p\ large)$.

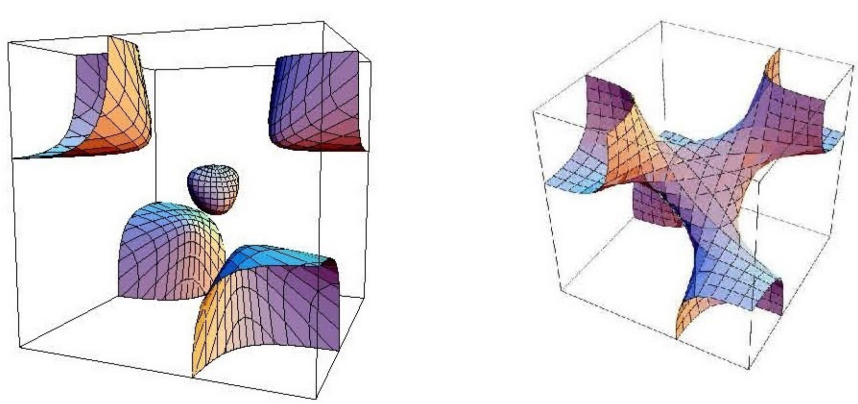
S_0 Markoff's cubic surface

S_4 Cayley's cubic surface

<u>$S_k(\mathbb{R})$ for different k:</u>



$k = 0$ and $k = 4$



$k = 2$ and $k = 8$

DIOPHANTINE ANALYSIS OF $X_k(\mathbb{Z})$:

PART I (JOINT WITH A. GHOSH)

• $k=0$, MARKOFF'S EQUATION

$X_k(\mathbb{Z})$ CONSISTS OF TWO

Γ ORBITS: $(0,0,0)=\Gamma.(0,0,0)$, $\Gamma.(3,3,3)$.

• $k=4$ (CAYLEY CUBIC) IS ESSENTIALLY LINEARIZABLE AND IS VERY DIFFERENT TO ANY OTHE k, WE OMIT IT.

USING Γ ONE CAN USE DESCENT TO SHOW THAT $h(k)$ THE NUMBER OF ORBITS OF Γ ON $X_k(\mathbb{Z})$ IS FINITE (MARKOFF, HURWITZ, MORDELL).

OUR FIRST INTEREST IS $h(k)=0$ THAT IS $X_k(\mathbb{Z})=\phi$.

GENERIC k'S:

THE CONGRUENCE OBSTRUCTIONS ARE mod 4, 9

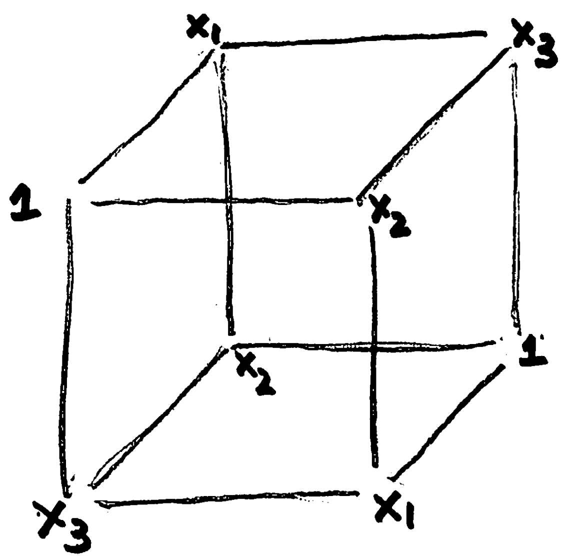
$k\equiv 3(4)$ AND $k\equiv 3,6\ (9)$ ARE IMPOSSIBLE.

WE AVOID THESE k'S AS WELL AS THOSE FOR WHICH THERE IS AN $x \in X_k(\mathbb{Z})$ WITH $x_j \in \{0, \pm 1, \pm 2\}$. THESE k'S ARE EXPLICIT AND CAN BE STUDIED AND ARE OF ZERO DENSITY. THE REMAINING k'S HAVE DENSITY

$$\#\{k \leq K : k \text{ GENERIC}\} \sim \frac{7}{12} K.$$

FOR THESE GENERIC k'S THERE ARE NO LOCAL OBSTUCTIONS AND $h(k) = 0$ IS A FAILURE OF THE HASSE PRINCIPLE.

- BHARGAVA CUBES AND MARKOFF



THREE SLICINGS GIVE

$$M_1 = \begin{bmatrix} 1 & x_2 \\ x_3 & x_1 \end{bmatrix}, N_1 = \begin{bmatrix} x_1 & x_3 \\ x_2 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & x_3 \\ x_1 & x_2 \end{bmatrix}, N_2 = \begin{bmatrix} x_2 & x_1 \\ x_3 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & x_1 \\ x_2 & x_3 \end{bmatrix}, N_3 = \begin{bmatrix} x_3 & x_1 \\ x_2 & 1 \end{bmatrix}$$

THESE GIVE THREE RECIPROCAL QUADRATIC FORMS

$$Q_1 = (x_2x_3 - x_1)u^2 + (1 + x_1^2 - x_2^2 - x_3^2)uv + (x_2x_3 - x_1)v^2$$
$$Q_2 = (x_1x_3 - x_2)u^2 + (1 + x_2^2 - x_1^2 - x_3^2)uv + (x_1x_3 - x_2)v^2$$
$$Q_3 = (x_1x_2 - x_3)u^2 + (1 + x_3^2 - x_1^2 - x_2^2)uv + (x_1x_2 - x_3)v^2$$

THEY HAVE COMMON DISCRIMINANT

$$\Delta(x_1, x_2, x_3) = (1 + x_1 + x_2 + x_3)(1 + x_2 - x_1 - x_3)(1 + x_3 - x_1 - x_2)(1 + x_1 - x_2 - x_3)$$

WHICH TURNS OUT TO BE AN "EXACT" GAUGE FUNCTION FOR DESCENT: UNDER R_1, R_2, R_3

$$\Delta_1(x) - \Delta(x) = x_2x_3(x_2x_3 - 2x_1)\left[2(k-5) + (x_2^2 - 4)(x_3^2 - 4)\right]$$
$$\Delta_2(x) - \Delta(x) = x_1x_3(x_1x_3 - 2x_2)\left[2(k-5) + (x_1^2 - 4)(x_3^2 - 4)\right]$$
$$\Delta_3(x) - \Delta(x) = x_1x_2(x_1x_2 - 2x_3)\left[2(k-5) + (x_1^2 - 4)(x_2^2 - 4)\right].$$

THIS LEADS TO "GHOSH REDUCED FORM"

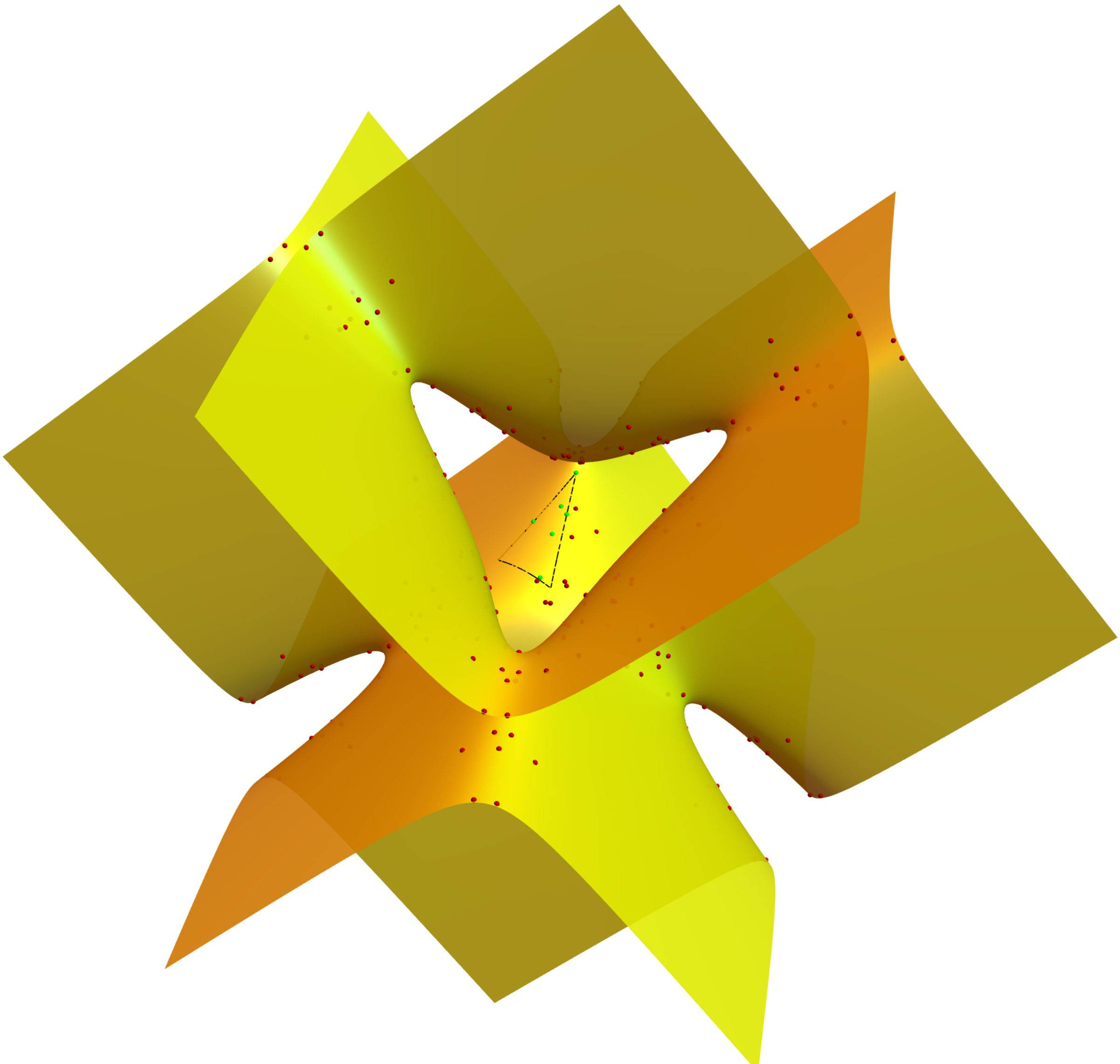
FOR $k \geq 5$ AND GENERIC.

$$\mathcal{F}_k = \{u \in \mathbb{R}^3 : \quad 3 \leq u_1 \leq u_2 \leq u_3 ; u_1^2 + u_2^2 + u_3^2 + u_1u_2u_3 \leq k\}$$

THEN EVERY POINT OF $\mathcal{F}_k(\mathbb{Z}) = \mathcal{F}_k \cap \mathbb{Z}^3$

CORRESPONDS TO A UNIQUE Γ-ORBIT OF $X_k(\mathbb{Z})$ WITH $(x_1, x_2, x_3) = (-u_1, u_2, u_3)$.

THIS ALLOWS FOR A NUMERICAL AND ANALYTIC STUDY OF $X_k(\mathbb{Z})$, $h(k$



LAST PICTURE IS FOR

$k = 3658$, $h(k) = 6$

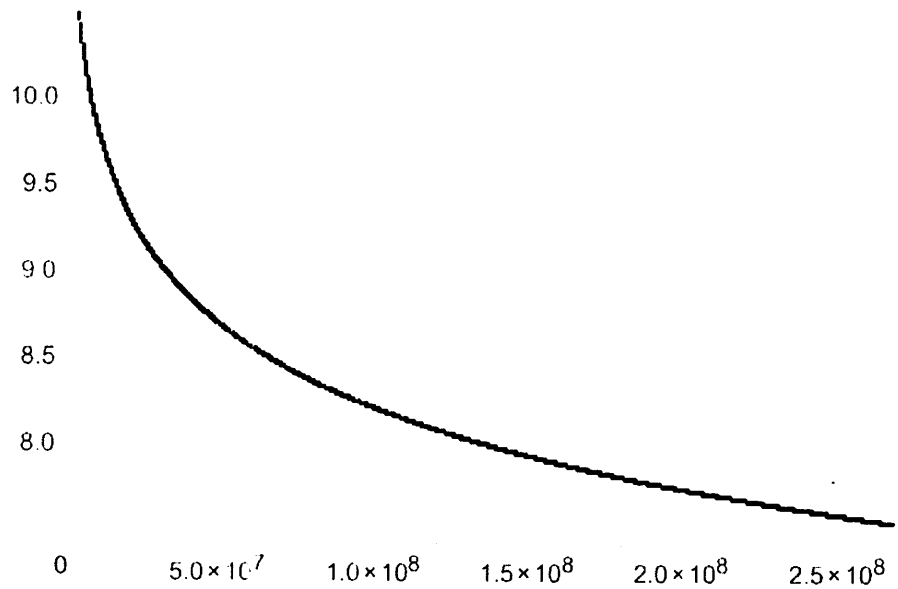


FIGURE 3. Percentages of Hasse failures.

NUMERICS SUGGEST THAT THE NUMBER OF HASSE FAILURES WITH $k \leq K$ IS $O(K^{\theta})$ WITH

$\theta \approx 0.88\ldots$

THEOREM 1:

$$\sum_{5 \le k \le K} h(k) \sim \frac{K(\log K)^2}{36} \quad \text{as } K \to \infty .$$

THEOREM 2:

THERE ARE INFINITELY MANY GENERIC k's (AT LEAST $K^{1/2}$ WITH $k \le K$) FOR WHICH $h(k)=0$, THAT IS HASSE FAILURES.

"THEOREM" 3

THERE ~~ARE~~ IS A POSITIVE PROPORTION OF (GENERIC) k's FOR WHICH $h(k) \neq 0$; THAT IS HASSE IS TRUE.

COMMENTS ON PROOFS:

(•) WITH THE EXPLICIT GHOSH REDUCTION, THEOREM 1 IS PROVEN BY THE USUAL INTEGER POINTS IN A TENTACLED REGION ARGUMENT.

(•) THEOREM 2 USES GLOBAL QUADRATIC RECIPROCITY TO DEFINE A "BRAUER-MANIN" TYPE OBSTRUCTION COMING FROM X_k mod p WITH $k \equiv 4\ (p)$ AND PROPERTIES OF THE CAYLEY CUBIC mod p, IT IS CLOSELY RELATED TO AN ARGUMENT OF MORDELL.

(•) THEOREM 3 IS BASED ON SOME TECHNIQUES OF BOURGAIN AND FUCHS IN THE STUDY OF CURVATURES IN INTEGRAL APOLLONIAN PACKINGS.

EXAMPLES OF HASSE FAILURES

$k = 4 + 2\nu^2$ WITH

ν HAVING ALL ITS PRIME FACTORS $\pm 1 \pmod 8$

AND $\nu \in \{0, \pm 3, \pm 4\} \pmod 9$.

DIOPHANTINE ANALYSIS OF $X_k(\mathbb{Z})$:

PART II (JOINT WITH BOURGAIN AND GAMBURD)

STRONG APPROXIMATION:

Once we have $\hat{x} \in X_k(\mathbb{Z})$ we get $\mathcal{O} = \Gamma\hat{x} \subset X_k(\mathbb{Z})$ which (except for very special k that are easily determined) is already Zariski dense in X_k, and one can ask for strong approximation.

That is whether for $q \geq 1$.

$$X_k(\mathbb{Z}) \xrightarrow{\bmod q} X_k(\mathbb{Z}/q\mathbb{Z}) \text{ is onto?}$$

Reducing the Γ action mod q gives a homomorphism

$$\Gamma \longrightarrow \text{Permutations}\left(X_k(\mathbb{Z}/q\mathbb{Z})\right),$$

and the question is how big is the image?

- If the permutation group is transitive on $X_k(\mathbb{Z}/q\mathbb{Z})$ (note $|X_k(\mathbb{Z}/q\mathbb{Z})| \approx q^2$) then we have strong approximation!

- What we can show (at least if $q = p$ a prime) is that the orbits are as large as possible and $X_k(\mathbb{Z})$ obeys a suitable form of strong approximation.

This is perhaps quite surprising since $X_k(\mathbb{Z})$ itself is a very sparse set; According to results of D. Zagier ($k=0$) and M. Mirzakhani in general (and more general character varieties)

$$\left| \{ x \in X_k(\mathbb{Z}) : |x| \leq T \} \right| \approx (\log T)^2$$

$(k \neq 4)$.

FINITE ORBITS OF Γ ON $A^3(\overline{\mathbb{Q}})$:

IN ORDER TO FORMULATE THE TRANSITIVITY PROPERTIES OF Γ WE NEED FIRST TO CLASSIFY THE FINITE ORBITS IN CHARACTERISTIC ZERO AS THESE OCCUR IN $\mathbb{Z}/p\mathbb{Z}$ FOR CERTAIN p'S.

THEOREM 4: THERE ARE FINITELY MANY FINITE Γ-ORBITS ON $X_k(\overline{\mathbb{Q}})$ AND THESE MAY BE DETERMINED EFFECTIVELY.

REMARKABLY THIS DETERMINATION HAS ALSO BEEN CARRIED OUT BY DUBROVIN/MAZZOCCO (AND LISOVYY/TYKHYY FOR THE 4-HOLED SPHERE) IN THEIR CLASSIFICATION OF THE PAINLEVÉ VI'S WHICH ARE ALGEBRAIC FUNCTIONS OF z.

$$\frac{d^2y}{dz^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)\left(\frac{dy}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)\frac{dy}{dz}$$
$$+ \frac{y(y-1)(y-z)}{z^2(1-z)^2}\left[\alpha + \frac{\beta z}{y^2} + \frac{\gamma(z-1)}{(y-1)^2} + \frac{\delta z(z-1)}{(y-z)^2}\right]$$

SUITABLE COORDINATES FOR THE SOLUTIONS YIELD THE MARKOFF SURFACES AND THE (NONLINEAR) MONODROMY GROUP CORRESPONDS TO Γ; MOREOVER FINITE ORBITS CORRESPOND TO ALGEBRAIC SOLUTIONS!.

MAIN CONJECTURE

X_k, Γ AS ABOVE. FOR p LARGE THE Γ-ORBITS IN $X_k(\mathbb{Z}/p\mathbb{Z})$ CONSIST OF THE (APRIORI DETERMINED) FINITELY MANY $X_k(\bar{\mathbb{Q}})$ ORBITS THAT OCCUR IN $X_k(\mathbb{Z}/p\mathbb{Z})$ AND THE COMPLEMENT OF THESE $X_k^*(\mathbb{Z}/p\mathbb{Z})$ WHICH IS A SINGLE (BIG) Γ-ORBIT $(k \neq 4)$.

EG: $k = 0$, MARKOFF EQUATION, THE ONLY FINITE $\bar{\mathbb{Q}}$ ORBIT IS $\{(0,0,0)\}$, SO THAT Γ ACTS TRASITIVELY ON $X_k^*(\mathbb{Z}/p\mathbb{Z})$.

THEOREM 5: (GIANT ORBIT):

FOR $\varepsilon > 0$ AND p LARGE THERE IS A Γ ORBIT $\mathcal{O}(p) \subset X^*(\mathbb{Z}/p\mathbb{Z})$ FOR WHICH

$$|X^*(\mathbb{Z}/p\mathbb{Z}) \setminus \mathcal{O}(p)| \ll p^{\varepsilon}$$

(note that $|X^*(\mathbb{Z}/p\mathbb{Z})| \sim p^2$),

AND EVERY Γ-ORBIT IN $X^*(\mathbb{Z}/p\mathbb{Z})$ HAS SIZE AT LEAST $(\log p)^{1/3}$.

WE CAN PROVE THE MAIN CONJECTURE AS LONG AS p^2-1 IS NOT VERY SMOOTH (THAT IS IT DOES NOT HAVE A VERY LARGE NUMBER OF SMALL FACTOR EG $p-1 = m!$ IS PROBLEMATIC.

VERY FEW PRIMES HAVE THIS HIGHLY SMOOTH PROPERTY AND HENCE WE PROVE THE MAIN CONJECTURE EXCEPT POSSIBLY FOR A SMALL SET OF PRIMES.

THEOREM 6: THE SET OF PRIMES E FOR WHICH THE MAIN CONJECTURE FAILS SATISFIES

$$|\{p \in E : p \leq T\}| \ll_{\varepsilon} T^{\varepsilon}, \text{ FOR } \varepsilon > 0.$$

THEOREM 7: (C. MEIRI AND D. PUDER) FOR $k=0$ (MARKOFF'S SURFACE) AND $p \equiv 1 (4)$ AND FOR WHICH Γ ACTS TRANSITIVELY ON $X_0^*(\mathbb{Z}/p\mathbb{Z})$, THE IMAGE OF Γ IN THE PERMUTATIONS OF $X_0^*(\mathbb{Z}/p\mathbb{Z})$ IS AS LARGE AS IT CAN BE (ALTERNATING OR SYMMETRIC GROUP).

MARKOFF NUMBERS: ($k=0$)

$$X : \quad x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0 \qquad (*)$$

- MARKOFF TRIPLES ARE SOLUTIONS TO (*) WITH $x_j \geq 1$ ($\Gamma.(1,1,1)$ GIVES ALL).
- MARKOFF NUMBERS M ARE NUMBERS WHICH ARE CO-ORDINATES OF MARKOFF TRIPLES.

OUR RESULTS GIVE (AT LEAST FOR THOSE p'S FOR WHICH TRANSITIVITY IS PROVED) THAT THE ONLY CONGRUENCES ON MARKOFF NUMBERS ARE THE 'OBVIOUS' ONES (FROBENIUS).

COUNTING(*): (ZAGIER , MIRZAKHANI)

$$\sum_{\substack{m \le T \\ m \in M}} 1 \sim c (\log T)^2 \quad , c \neq 0 .$$

THEOREM 8: ALMOST ALL MARKOFF NUMBERS ARE COMPOSITE;

$$\sum_{\substack{p \le T \\ p \text{ prime} \\ p \in M}} 1 = o\Big(\sum_{\substack{m \le T \\ m \in M}} 1 \Big), \text{ AS } T \to \infty .$$

COMMENTS ON PROOFS:

THE 'DEHN TWISTS' D_1, D_2, D_3 IN Γ

$$D_1 : (x_1, x_2, x_3) \longrightarrow (x_1, x_1 x_2 - x_3, x_2)$$

AND SIMILARLY FOR D_2, D_3

PRESERVE THE CONIC SECTIONS OF X_k.

$$X_k \cap \text{PLANE } x_1 = \hat{x}_1$$

AND $D_1^j \xi$, $j = 1, 2, \ldots$

SWEEPS OUT A SUBSET OF THE CONIC SECTION. IF THE ORDER OF THE ROTATION D_1 ACTING IN THIS PLANE IS OPTIMALLY IN THE $\mathbb{F}_p$ SETTING (AND INFINITE IN THE $\bar{\mathbb{Q}}$ SETTING) THEN THE Γ ORBIT OF ξ CONTAINS THIS ENTIRE CONIC SECTION. SO THE BASIC IDEA TO SHOW THAT ONE CAN INCREASE THE ORDERS OF THESE DEHN TWISTS BY MOVING TO POINTS IN ITS ORBIT. DOING SO REPEATEDLY EVENTUALLY LEADS TO CONNECTING EVERY $\xi \in X_k^*(\mathbb{Z}/p\mathbb{Z})$ TO THE GIANT ORBIT.

A KEY PROBLEM THAT INTERVENES IS TO GIVE AN UPPER BOUND TO THE NUMBER OF SOLUTIONS TO

$$\xi + \frac{b}{\xi} = \eta + \frac{1}{\eta} \quad , b \neq 1$$

$\xi \in H_1$, $\eta \in H_2$, $|H_2| \leq |H_1| \leq p^{1-\varepsilon}$

WITH H_1, H_2 SUBGROUPS OF $\mathbb{F}_p^*$ (OR $\mathbb{F}_{p^2}^*$). THE TRIVIAL UPPER BOUND IS $2|H_2|$ AND WE SEEK A BOUND

$$\ll |H_1|^\tau \quad \text{WITH} \quad \tau < 1 .$$

IF $|H_1| \geq p^{1/2}$ ONE CAN PROCEED USING WEIL'S RIEMANN HYPOTHESIS FOR CURVES OVER FINITE FIELDS.

IF $|H_1|$ IS SMALLER THIS IS OF NO USE AND WE OBTAIN THE REQUISITE BOUND BY ONE OF TWO METHODS

(A) STEPANOV'S AUXILIARY POLYNOMIAL METHOD

(B) BOURGAINS PROJECTIVE SZEMEREDI-TROTTER THEOREM OVER $\mathbb{F}_p$.

FOR THE FINITE $\bar{\mathbb{Q}}$ ORBITS OF Γ WE USE THE SAME DEHN TWISTS WHOSE ORDERS MUST BE FINITE. THIS LEADS TO THE EQUATION

$$(\lambda_1+\lambda_1^{-1})^2+(\lambda_2+\lambda_2^{-1})^2+(\lambda_3+\lambda_3^{-1})^2-(\lambda_1+\lambda_1^{-1})(\lambda_2+\lambda_2^{-1})(\lambda_3+\lambda_3^{-1}) = k$$

TO BE SOLVED WITH $\lambda_1, \lambda_2, \lambda_3$ IN ROOTS OF UNITY.

LANG'S $\mathbb{G}_M$ CONJECTURE AND ITS EFFECTIVE SOLUTIONS ALLOW ONE TO FIND THE (TYPICALLY) FINITELY MANY SOLUTIONS. THE FINITE Γ-ORBITS ARE THEN RESTRICTED TO LIE IN THESE FINITE SETS.

AS WE NOTED THE MARKOFF SURFACES ARE JUST THE FIRST OF THE AFFINE CHARACTER VARIETIES FOR WHICH THE MAPPING CLASS GROUP IS A POWERFULL TOOL FOR DESCENT AND DIOPHANTINE ANALYSIS. FOR $\pi_1(\Sigma_{g,n})$ PETER WHANG HAS MADE A SIGNIFICANT START IN THIS STUDY.