Dynamical properties of weak model sets

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joint work with Gerhard Keller, Erlangen

squarefree integers $\mathcal{S}$ as a cut-and-project set

(cf. Meyer 73, Baake–Moody–Pleasants 99, Sing 05, ...)

- $n \in \mathbb{Z}$ squarefree $\iff n \mod p^2 \neq 0$ for all primes p
- consider the compact product group $H := \prod_p (\mathbb{Z}/p^2\mathbb{Z})$
- dense embedding of $\mathbb{Z}$ into H by CRT:

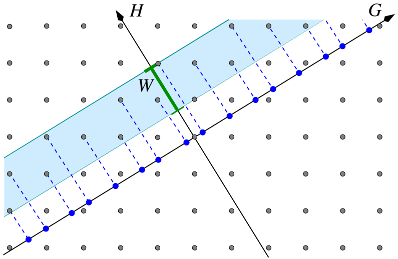
$$n \mapsto \iota(n) = (n \mod p^2)_p,$$

- write $G := \mathbb{Z}$ and note $\mathcal{L} := \{(n, \iota(n)) : n \in \mathbb{Z}\}$ is lattice in $G \times H$
- lattice $\mathcal{L}$ has compact torus $\hat{X} := (G \times H)/\mathcal{L} \sim H$
- $\mathcal{S} \subseteq G$ given by some "cut-and-project construction"

$$\mathcal{S} = \pi^G(\mathcal{L} \cap (G \times W)), \qquad W = \prod_p (\mathbb{Z}/p^2\mathbb{Z}) \backslash \{0_p\}$$

- analyse squarefree flow $\overline{\{g\mathcal{S} : g \in G\}}$ by first studying associated dynamics on $G \times H$ and then projecting to G

cut-and-project schemes and weak model sets

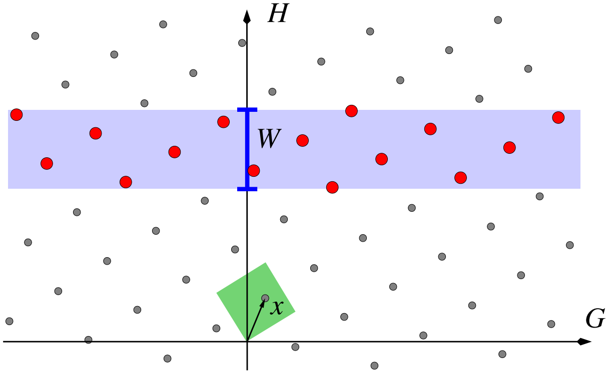


- *G physical*, *H internal space*, LCSCA groups, $\mathcal{L} \subseteq G \times H$ lattice
- infinite strip parallel to G defined by compact (!) *window* $W \subset H$
- *weak model set* by projecting lattice points inside strip to G
- assume wlog that projection of $\mathcal{L}$ is dense in H
- assume that distinct lattice points have distinct G-projection

some history

- introduced by Meyer, Schreiber in the 70's in the context of commutative harmonic analysis
- re-discovered in about 1984 for describing physical quasicrystals (Kramer, Levine–Steinhardt, Katz–Duneau, ...)
- dynamical properties were studied since about 1995 (Radin, Robinson, Moody, Baake, Lenz, Kellendonk, Arnoux, ...)
- mainly for "simple" windows ($\mathrm{int}(W) \neq \varnothing$, $m_H(\partial W) = 0$), uniquely ergodic dynamical systems of topological entropy 0
- squarefree integers and visible lattice points: $W = \partial W$, positive topological entropy (Baake–Moody–Pleasants 1999)
- recent interest due to $\mathcal{B}$-free systems and connections to Sarnak's program

torus parametrisation of weak model sets



green: FD of torus $\hat{X}$, red: $(x + \mathcal{L}) \cap (G \times W) = \mathrm{supp}(\nu_W(\hat{x}))$

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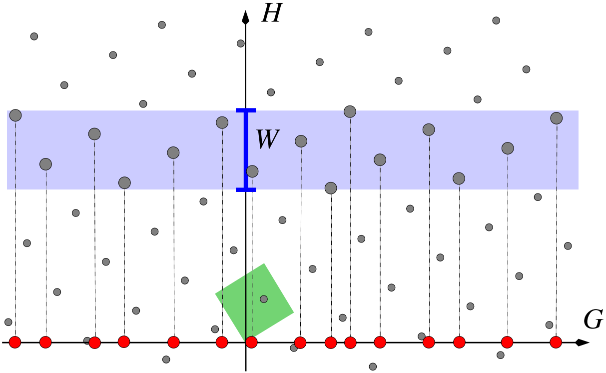

green: FD of torus $\hat{X}$, red: $\pi^G((x+\mathcal{L}) \cap (G \times W)) = \mathrm{supp}(\nu_W^G(\hat{x}))$

configurations and associated dynamical systems

configurations

- configuration $\nu_W(\hat{x}) \in \mathcal{M}$
  (locally finite measures on $G \times H$, vague topology)
- parametrised by torus points $\hat{x} = x + \mathcal{L} \in \hat{X} = (G \times H)/\mathcal{L}$
- projected measure $\nu_W^G(\hat{x}) = \pi_*^G \circ \nu_W(\hat{x}) \in \mathcal{M}^G$
  (locally finite measures on G, vague topology)

relevant topological dynamical systems (with induced G-action S)

- $\mathcal{M}_W = \overline{\nu_W(\hat{X})} \subseteq \mathcal{M}$ configurations on $G \times H$
- $\mathcal{M}_W^G = \overline{\nu_W^G(\hat{X})} \subseteq \mathcal{M}^G$ configurations on G
- hull $\mathcal{M}_W^G(\hat{x}) \subseteq \mathcal{M}_W^G$: orbit closure of configuration $\nu_W^G(\hat{x})$

Consider the map $\nu_W : \hat{X} \to \mathcal{M}_W$. It gives rise to even larger dynamical systems via its graph closure.

graph dynamical systems

consider the topological dynamical system

$$\overline{\mathrm{graph}(\nu_W)} = \overline{\{(\hat{x}, \nu_W(\hat{x})) : \hat{x} \in \hat{X}\}} \subseteq \hat{X} \times \mathcal{M}_W$$

with induced G-action $\hat{T} \times S$

- $\nu_W : \hat{X} \to \mathcal{M}_W$ upper semicontinuous as W compact

$$\hat{x}_n \to \hat{x} \quad \Longrightarrow \quad \nu \leqslant \nu_W(\hat{x}) \text{ for any vague limit point } \nu \text{ of } (\nu_W(\hat{x}_n))_n$$

- continuity points $C_W \subseteq \hat{X}$ of ν_W are dense G_δ in $\hat{X}$:

$$(g, h) + \mathcal{L} \in C_W \iff h + \ell_H \notin \partial W \text{ for every } \ell \in \mathcal{L}$$
$$\iff h \in \bigcap_{\ell \in \mathcal{L}} ((\partial W)^c - \ell_H)$$

- $C_W \subseteq \hat{X}$ is $\hat{T}$-invariant

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- $C_W \subseteq \hat{X}$ is $\hat{T}$-invariant

topological results for configurations on $G \times H$

Theorem

a) $\overline{\mathrm{graph}(\nu_W|_{C_W})}$ *is the only minimal subset of* $\overline{\mathrm{graph}(\nu_W)}$.

b) $\overline{\nu_W(C_W)}$ *is the only minimal subset of* $\mathcal{M}_W = \overline{\nu_W(\hat{X})}$.

Proof.

a) $\varnothing \neq A \subseteq \overline{\mathrm{graph}(\nu_W)}$ closed invariant $\Rightarrow \varnothing \neq \pi^{\hat{X}}(A) \subseteq \hat{X}$ closed invariant $\Rightarrow \pi^{\hat{X}}(A) = \hat{X} \supseteq C_W$, since $(\hat{X}, \hat{T})$ minimal $\Rightarrow$ $A \supseteq \mathrm{graph}(\nu_W|_{C_W}) \Rightarrow A \supseteq \overline{\mathrm{graph}(\nu_W|_{C_W})} =: A_{min}$

b) $\varnothing \neq B \subseteq \mathcal{M}_W$ closed invariant $\Rightarrow \varnothing \neq (\pi_*^{G\times H})^{-1}(B) \subseteq \overline{\mathrm{graph}(\nu_W)}$ closed invariant $\Rightarrow (\pi_*^{G\times H})^{-1}(B) \supseteq A_{min} \Rightarrow B \supseteq \pi_*^{G\times H}(A_{min}) = \overline{\nu_W(C_W)}$ □

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injectivity properties of $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$

- complicated as W interfers with $\pi^H(\mathcal{L})$, which is dense in H
- for $A \subseteq H$ one says
  - A *aperiodic* if $h + A = A$ implies $h = 0$
  - A *topologically regular* if $A = \overline{\mathrm{int}(A)}$

Lemma

If W is topologically regular and aperiodic, then $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is a homeomorphism.

- above results transfer to G, if W is aperiodic and top regular
- previous dynamical results often assumed an aperiodic and topologically regular window
- we are interested in results beyond this case

injectivity of π_*^G and configuration windows

consider the map $\mathcal{S}_H : \mathcal{M}_W \to \mathcal{K}(W)$ (cpct subsets of W) given by

$$\mathcal{S}_H(\nu) = \overline{\pi^H(\mathrm{supp}(\nu))} = \mathrm{supp}(\pi_*^H \nu)$$

How "small" can $\mathcal{S}_H(\nu)$ be?

- $\mathcal{S}_H(\nu_W(\hat{x})) = \overline{\pi^H(x + \mathcal{L}) \cap \mathrm{int}(W)} = \overline{\mathrm{int}(W)}$ for $\hat{x} \in C_W$
- this implies $\overline{\mathrm{int}(W)} \subseteq \mathcal{S}_H(\nu) \subseteq W$ for any $\nu \in \mathcal{M}_W$
  (as $\overline{\nu_W(C_W)} \subseteq \overline{\{S_g \nu : g \in G\}}$ and $\mathcal{S}_H$ is lower semicontinuous)

aperiodicity of $\mathrm{int}(W)$ implies that $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is $1-1$:

- assume $\nu, \nu' \in \mathcal{M}_W$ are such that $\pi_*^G(\nu) = \pi_*^G(\nu')$
- then ν' and ν can only differ by some overall shift $d \in H$
- hence also $d + \mathcal{S}_H(\nu) = \mathcal{S}_H(\nu')$ and thus $d + \mathrm{int}(W) = \mathrm{int}(W)$

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topological results for configurations on G

Theorem

Assume that $\mathrm{int}(W)$ *is aperiodic (so in particular non-empty).*

a) $(\mathcal{M}_W, S)$ *and* $(\mathcal{M}_W^G, S)$ *are isomorphic, and both are almost* $1-1$ *extensions of their maximal equicontinuous factor* $(\hat{X}, \hat{T})$.

b) *Denote by* $\Gamma : \mathcal{M}_W^G \to \hat{X}$ *the above factor map. If* M *is a non-empty, closed* S*-invariant subset of* $\mathcal{M}_W^G$*, then* (M, S) *is an almost* $1-1$ *extension of its maximal equicontinuous factor* $(\hat{X}, \hat{T})$ *with factor map* $\Gamma|_M$.

remark:

- $\mathrm{int}(W) \neq \varnothing$ aperiodic if H only has trivial compact subgroups

a) $\pi^{\hat{X}} : (\overline{\mathrm{graph}(\nu_W)}, S) \to (\hat{X}, \hat{T})$ is a $1-1$ extension of the equicontinuous factor $(\hat{X}, \hat{T})$, as C_W is dense G_δ. By a factorisation argument, this also holds for the maximal equicontinuous factor, which must then coincide with $(\hat{X}, \hat{T})$.

$\pi_*^{G\times H} : \overline{\mathrm{graph}(\nu_W)} \to \mathcal{M}_W$ is $1-1$ on non-zero configurations, as the torus coordinate can then be uniquely reconstructed. The condition $\mathrm{int}(W) \neq \varnothing$ excludes zero configurations.

$\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is $1-1$ if $\mathrm{int}(W)$ is aperiodic (see above)

b) follows with a)

topological results for configurations on G

- In general, the MECF is some proper factor group of $\hat{X}$.
- For $A \subseteq H$, consider its period group

$$H_A := \{h \in H : h + A = A\}, \qquad \mathcal{H}_A = \{0\} \times H_A \subseteq G \times H$$

Theorem

Assume that $\mathrm{int}(W) \neq \varnothing$*. Let* $\widehat{X'} = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{\mathrm{int}(W)})$ *with induced* G*-action* $\widehat{T'}$*. Let* M *be any non-empty, closed* S*-invariant subset of* $\mathcal{M}_W^G$.

a) $(\widehat{X'}, \widehat{T'})$ *is the maximal equicontinuous factor of the topological dynamical system* (M, S).

b) *If* $H_{\mathrm{int}(W)} = H_W$*, then* (M, S) *is an almost* $1-1$ *extension of* $(\widehat{X'}, \widehat{T'})$.

□

topological results

- The minimal subsystem may be trivial.
- $\text{int}(W) = \varnothing$ iff $\overline{\nu_W(C_W)} = \{\underline{0}\}$ iff $(\mathcal{M}_W, S)$ has a trivial maximal equicontinuous factor:

$$\begin{aligned}\text{int}(W) = \varnothing &\Leftrightarrow W = \partial W \\ &\Leftrightarrow C_W = \pi^{\hat{X}}((\pi^H)^{-1}(\bigcap_{\ell \in \mathcal{L}} (W^c - \ell_H))) \\ &\Leftrightarrow C_W = \pi^{\hat{X}}(x \in G \times H : (x + \mathcal{L}) \cap (G \times W) = \varnothing) \\ &\Leftrightarrow \nu_W(C_W) = \{\underline{0}\}\end{aligned}$$

- above we considered the case $\text{int}(W) \neq \varnothing$
- if $\text{int}(W) = \varnothing$, analogous results hold for the maximal equicontinuous *generic* factor (Keller 2016)

Mirsky measure for configurations on $G \times H$

Mirsky measure $Q_W := m_{\hat{X}} \circ (\nu_W)^{-1}$ is lift of Haar measure $m_{\hat{X}}$ on $\hat{X}$

- was studied in the squarefree case first by Mirsky
- only invariant probability measure on $(\mathcal{M}_W, S)$ if $m_H(\partial W) = 0 \Leftrightarrow m_{\hat{X}}(C_W) = 1$

Proposition

Assume that $m_H(W) > 0$. Then $(\mathcal{M}_W, Q_W, S)$ is measure-theoretically isomorphic to $(\hat{X}, m_{\hat{X}}, \hat{T})$. □

reason:

- $\nu_W : \hat{X} \to \mathcal{M}_W$ provides measure-theoretic factor map
- "shift vector map" $\widehat{\pi} : \mathcal{M}_W \backslash \{\underline{0}\} \to \hat{X}$ gives (continuous) measure-theoretic factor map, as $m_H(W) > 0$ implies $Q_W(\{\underline{0}\}) = 0$
- note $\widehat{\pi} \circ \nu_W = id$ whenever composition is well defined

injectivity properties of $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$

- int(W) aperiodic useless in measurable context, if int$(W) = \varnothing$
- let P be an ergodic S-invariant propability measure on $\mathcal{M}_W$
- there is $W_P \subseteq W$ such that for P-aa $\nu \in \mathcal{M}_W$

$$\mathcal{S}_H(\nu) = W_P$$

- hence π_*^G is $1-1$ on $(\mathcal{S}_H)^{-1}\{W_P\} \subseteq \mathcal{M}_W$ if W_P aperiodic
- in fact W_P is *Haar regular*, i.e., for any open $U \subseteq H$

$$(U \cap W_P) \neq \varnothing \implies m_H(U \cap W_P) > 0$$

Mirsky measure for configurations on G

Mirsky measure $Q_W^G := m_{\hat{X}} \circ (\nu_W^G)^{-1}$ is lift of Haar measure $m_{\hat{X}}$ on $\hat{X}$

- above results can be transferred if projection $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is $1-1$ on a subset of Q_W-measure 1
- condition W_{Q_W} aperiodic is equivalent to *Haar aperiodicity* of W

$$m_H((h+W)\Delta W) = 0 \Longrightarrow h = 0$$

- W Haar aperiodic implies $m_H(W) > 0$.

Theorem

Suppose that W is Haar aperiodic. Then $(\mathcal{M}_W^G, Q_W^G, S)$ is measure-theoretically isomorphic to $(\hat{X}, m_{\hat{X}}, \hat{T})$. □

Mirsky measure for configurations on G

- In general, there is an isomorphism to a factor group of $\hat{X}$.
- consider the group H_W^m of Haar periods of W, i.e.

$$H_W^m = \{h \in H : m_H((h+W)\Delta W) = 0\}, \qquad \mathcal{H}_W^m = \{0\} \times H_W^m \subseteq G \times H$$

Theorem

Suppose $m_H(W) > 0$. Let $\widehat{X'} = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_W^m)$ with induced G-action $\widehat{T'}$ and Haar measure $m_{\widehat{X'}}$. Then $(\mathcal{M}_W^G, Q_W^G, S)$ is measure-theoretically isomorphic to $(\widehat{X'}, m_{\widehat{X'}}, \widehat{T'})$. □

Mirsky measure $Q_W = m_{\hat{X}} \circ (\nu_W)^{-1}$ and maximal density

- Fix tempered van Hove sequence (e.g. centred n-balls in $G = \mathbb{R}^d$)
- We say that ν has *maximal density* if

$$\lim_{n\to\infty} \frac{\nu(A_n \times H)}{m_G(A_n)} = \mathrm{dens}(\mathcal{L}) \cdot m_H(W)$$

  (upper density $\leqslant$ always true)
- consider the set $\mathcal{M}'_W \subseteq \mathcal{M}_W$ of maximal density configurations
- $Q_W(\mathcal{M}'_W) = 1$, i.e., maximal density is Q_W-generic (Moody 2002)
- $\pi^G_*|_{\mathcal{M}'_W} : \mathcal{M}'_W \to \mathcal{M}^G_W$ is one-to-one, if W_{Q_W} is aperiodic
- true as for $\nu \in \mathcal{M}'_W$ we always have

$$W_{Q_W} \subseteq \mathcal{S}_H(\nu) \subseteq W$$

Mirsky measure on configuration hulls in G

For $\hat{x} \in \hat{X}$, consider the orbit closure $\mathcal{M}_W^G(\hat{x}) \subseteq \mathcal{M}_W^G$ of the configuration $\nu_W^G(\hat{x})$ under the action of S.

- if $\nu_W^G(\hat{x})$ has maximal density, then $\mathrm{supp}(Q_W^G) \subseteq \mathcal{M}_W^G(\hat{x}) \subseteq \mathcal{M}_W^G$
- if in addition $\nu_W^G(\hat{x}) \in \mathrm{supp}(Q_W^G)$, then $\mathcal{M}_W^G(\hat{x}) = \mathcal{M}_W^G$
- these two conditions are generic for the Mirsky measure
- in that case, the above results apply to $\mathcal{M}_W^G(\hat{x})$ replacing $\mathcal{M}_W^G$
- in many examples Q_W^G has full support $\mathcal{M}_W^G$

measures for configurations on $G \times H$

setting

- $(\mathcal{M}_W, P, S)$ with P an ergodic S-invariant probability measure
- argue as in the Mirsky measure case

Proposition

If $m_H(W) > 0$ and if P is any S-invariant probability measure on $\mathcal{M}_W$, then $(\mathcal{M}_W, P, S)$ is a measure-theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$. □

measures for configurations on G

transfer above result to measures on $\mathcal{M}_W^G$

- for given ergodic S-invariant P^G on $\mathcal{M}_W^G$, there exists ergodic S-invariant P on $\mathcal{M}_W$ such that $P^G = P \circ (\pi_*^G)^{-1}$
- argue then as in Mirsky measure case

Theorem

Let P^G, P be ergodic S-invariant probability measures on $\mathcal{M}_W^G, \mathcal{M}_W$ such that $P^G = P \circ (\pi_*^G)^{-1}$, and suppose that W_P is aperiodic.
Then $(\mathcal{M}_W, P, S)$ and $(\mathcal{M}_W^G, P^G, S)$ are measure-theoretically isomorphic via π_*^G, and $(\mathcal{M}_W^G, P^G, S)$ is a measure-theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$. □

measures for configurations on G

- in general isomorphism to some proper factor group of $\hat{X}$
- consider the period group H_{W_P} of W_P

Theorem

Assume that $m_H(W) > 0$. Let P^G be an ergodic S-invariant probability measure on $\mathcal{M}_W^G \backslash \{\underline{0}\}$. Take any ergodic S-invariant probability measure P on $\mathcal{M}_W$ satisfying $P^G = P \circ (\pi_*^G)^{-1}$. Let $\widehat{X'} = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{W_P})$ with induced G-action $\widehat{T'}$ and Haar measure $m_{\widehat{X'}}$.
Then $(\mathcal{M}_W^G, P^G, S)$ is a measure theoretic extension of $(\widehat{X'}, m_{\widehat{X'}}, \widehat{T'})$.