Counting solutions in Intrinsc Diophantine approximation

December 15, 2016

Lumini conference: Ergodic theory and its connection to number theory and combinatorics

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Based on joint work with

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Dirichlet’s theorem in Euclidean space

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Intrinsic Diophantine approximation
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and call \( \mathbf{x} \) \( \psi \)-approximable if there are infinitely many solutions \((p, q) \in \mathbb{Z}^d \times \mathbb{N}_+\) to the Diophantine inequality:

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**Dirichlet’s theorem.** For \( \psi(q) = \frac{1}{q^{1/m}} \), every \( \mathbf{x} \in \mathbb{R}^d \) is \( \psi \)-approximable.
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Since the measure of the cube with edge size $2\psi(q)$ is $(2\psi(q))^d$, it follows that if the sum over all $\left(\begin{array}{c} p \\ q \end{array}\right) \in \mathbb{Z}^d \times \mathbb{N}^+$ with $p/q \in (0,1)^d$ of the measures of the cubes $B(p/q, \psi(q))$ is finite, or equivalently if $\sum_{q \geq 1} (2\psi(q))^d < \infty$, then the Borel-Cantelli lemma shows that almost every $x$ is not $\psi$-approximable.
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then the Borel-Cantelli lemma shows that almost every $x$ is not $\psi$-approximable.
The converse is given by **Khinchin’s theorem**: 

\[ \sum_{q} q \geq 1 \psi(q) \quad \text{and} \quad d = \infty, \]

then for almost every \( x \in \mathbb{R}^d \) the system of inequalities

\[ \| x - p/q \| \leq \psi(q) q \]

has infinitely many solutions, namely \( x \) is \( \psi \)-approximable.

This sharp dichotomy, giving rise to infinitely many solutions in the divergence case, is very satisfying. Naturally, it raises the following question: If \( x \) is \( \psi \)-approximable, so that there are infinitely many solutions to the Diophantine inequality, how many solutions are there of a given bounded size?
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Questions:
Khinchin’s theorem in Euclidean space (1924)

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- **Question:** If $x$ is $\psi$-approximable, so that there are infinitely many solutions to the Diophantine inequality, how many solutions are there of a given bounded size?
W. Schmidt’s theorem in Euclidean space (1960)

- Define the solution counting function at \( x \) (with gauge \( \psi \))

\[
N_T(x) = \left| \left\{ (p, q) \in \mathbb{Z}^d \times \mathbb{N}_+ ; 1 \leq q \leq T \text{ and } \left\| x - \frac{p}{q} \right\| \leq \frac{\psi(q)}{q} \right\} \right|
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W. Schmidt’s theorem: If \( V_T \to \infty \), namely if the divergence case of Khinchin’s theorem holds, then for almost every \( x \in \mathbb{R}^d \)

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N_T(x) = V_T + O_{x, \epsilon}(V_1^{2/3} + \epsilon T)
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and the following volume growth function

$$V_T = \sum_{(p,q) \in \mathbb{Z}^d \times N_+ , 1 \leq q \leq T , \frac{p}{q} \in (0,1)^d} m \left( B \left( \frac{p}{q}, \frac{\psi(q)}{q} \right) \right) = \sum_{1 \leq q \leq T} (2\psi(q))^d.$$
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(for all \(\epsilon > 0\)).
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We emphasize that the approximation process here is allowed to utilize all rational points in $\mathbb{R}^d$. 
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Previously, the problem of intrinsic Diophantine approximation has been studied mainly when the variety in question is a commutative algebraic group or an Abelian variety.
We will consider the problem of intrinsic Diophantine approximation on a homogeneous algebraic variety $X$ defined over $\mathbb{Q}$. For simplicity, we will assume it is simply transitive under an action of a simple algebraic group $G$ defined over $\mathbb{Q}$, which is algebraically simply connected.
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Denote by $X(\mathbb{R}) \subset \mathbb{R}^N$ the set of real solutions, and by $X(\mathbb{Q})$ the set of rational solutions. We would like to analyze the system of intrinsic Diophantine inequalities:

$$\|x - r\| \leq \epsilon \quad \text{and} \quad D(r) \leq \epsilon^{-\kappa}$$

with $x \in X(\mathbb{R})$ and $r \in X(\mathbb{Q})$, and $D(r)$ the reduced denominator.
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Our goal is to establish a value of $\kappa$ that gives an exponent of Diophantine approximation, to establish the existence of infinitely many solutions almost surely as in Khintchin’s theorem, and then to count their number, as in Schmidt’s theorem.
In our exposition here, we focus on the problem of intrinsic Diophantine approximation by a constrained set of rational points, namely those in $X(\mathbb{Z}[1/p])$ where $p$ is prime.
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Some examples

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Intrinsic Diophantine approximation
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- and we attach to \( X, \, G \) and \( p \) two parameters, as follows.
Growth parameter. For a general algebraic variety $X$, we consider the empirical distribution of $X(\mathbb{Z}[\frac{1}{p}])$ points, and set:

$$a_p(X) = \sup_{\text{compact } \Omega \subset X(\mathbb{R})} \limsup_{R \to \infty} \frac{\log |\{ r \in \Omega \cap X(\mathbb{Z}[\frac{1}{p}]) : D(r) \leq R \}|}{\log(R)}.$$
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**Spectral parameter.** $q_p(G)$ will denote the integrability parameter of spherical functions appearing in a suitable automorphic representation associated with the group $G$, as will be explained below.
We can now state:

- for almost every $x \in X(\mathbb{R})$, every $\kappa > \frac{\dim G}{a_p(G)} \frac{q_p(G)}{2}$, and $\epsilon \in (0, \epsilon_0(x, \kappa))$, the system of inequalities

  $$\|x - r\| \leq \epsilon \quad \text{and} \quad D(r) \leq \epsilon^{-\kappa}$$

has a solution $r \in X(\mathbb{Z}[1/p])$. 

Intrinsic Diophantine approximation
Uniform and almost sure approximation exponents on homogeneous varieties

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Infinitely many solutions almost surely

The value \( \tau = \dim G^q_p(G) \) provides an analog of Khinchin's theorem. Namely, for any rate function \( \psi \) the Diophantine inequality:

\[ \| x - r \| \leq \psi(D(r)) \]

has infinitely many solutions if

\[ \sum_{r \in X(Z[1/p])} \psi(D(r)) \tau + \epsilon = \infty \text{ (for some } \epsilon > 0) \]

Whereas conversely, if

\[ \sum_{r \in X(Z[1/p])} \psi(D(r)) \dim G < \infty \],

the inequality has finitely many solutions.

Note that usually \( q_p(G) \geq 2 \), and equality provides the best possible result, namely a sharp threshold in Khinchin's theorem. This occurs, for example, for \( Z[1/p] \)-approximations on \( S^2 \) and \( S^3 \).

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Note that usually \( q_p(G) \geq 2 \), and equality provides the best possible result, namely a sharp threshold in Khinchin’s theorem. This occurs, for example, for \( \mathbb{Z}[\frac{1}{p}] \)-approximations on \( S^2 \) and \( S^3 \).
Define the solution counting function (with gauge $\psi(h) = h^{-b}$ for suitable $b > 0$):

$$N_T(x) = \left| \left\{ r \in X(\mathbb{Z}[\frac{1}{p}]) ; 1 \leq D(r) \leq T, \text{ and } \|x - r\| < \psi(D(r)) \right\} \right|$$
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and the volume growth function:

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then there exist $\theta = \theta(b) \in (0, 1)$ such that for every $x \in X(\mathbb{R})$

$$N_T(x) = V_T + O_x \left( V_T^\theta \right)$$
Discrepancy of $\mathbb{Z} \left[ \frac{1}{p} \right]$ points on the variety

• Consider $\mathbb{Z} \left[ \frac{1}{p} \right]$-points on $X(\mathbb{R})$ with bounded denominator:

$$B_h = \left\{ r \in X(\mathbb{Z} \left[ \frac{1}{p} \right]) ; D(r) \leq p^h \right\}$$
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- Define the discrepancy of the family of finite sets $B_h$ in a set $B$ via
  \[ \Delta(B_h, B) = \left| \frac{B_h \cap B}{B_h} - m(B) \right|. \]
Discrepancy of $\mathbb{Z}[\frac{1}{p}]$ points on the variety

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- The discrepancy measures the deviation of the sets $\mathcal{B}_h$ from being fairly deposited in the set $B$. Taking $B = B(x, \delta)$ we are measuring the discrepancy at scale $\delta$, and would like to a bound valid also at very small scales $\delta \to 0$. 

Intrinsic Diophantine approximation
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- The uniform analog of Schmidt’s theorem stated above obviously yields an effective uniform bound on the discrepancy of $\mathbb{Z}[\frac{1}{p}]$ points on the variety, for suitable scales $\delta \sim h^{-b}$ for $0 < b < b_0$. 

Intrinsic Diophantine approximation
Some previous results

- Uniform discrepancy bounds on the spheres $S^2$ and $S^3$ for approximation by $\mathbb{Z}[\frac{1}{p}]$-points appears in the celebrated work of Lubotzky-Phillips-Sarank. Diophantine exponents can also be derived from their work, and in higher dimensional spheres from the work of Clozel, Ullmo and Oh. The intersection with group varieties amounts to $S^3$ only.
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- For approximation by all rational points, Kleinbock and Merrill established the best possible exponent in Dirichlet's theorem for uniform approximation on the spheres $S^d$, $d \geq 2$. They also obtain the sharp threshold in Khinchin's theorem.

- The methods they use involve homogeneous dynamics, particularly Dani-Margulis arguments regarding visiting times to shrinking neighborhoods of cusps, and a method previously introduced by Drutu to reduce the problem to this context. It is not clear whether this approach can yield an effective analog of Schmidt's theorem.
Scope of the method

It is possible to derive similar results on uniform and almost sure Diophantine approximation:

- For arbitrary simply transitive affine homogenous varieties of all semisimple groups, defined over an arbitrary number field $K$,
- Using $K$-rational points constrained by arbitrarily prescribed integrality conditions,
- Achieving simultaneous approximation over several completions of the field $K$,
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- Reduction of the lattice point counting problem to an effective mean ergodic theorem for the action of the group on the probability space $G/\Gamma$.

- Utilizing spectral estimates in the automorphic representation of $G$ in $L^2(G/\Gamma)$ to bound the averages appearing in the mean ergodic theorem.
In order to study the density of the set $X(\mathbb{Z}[1/p])$ in $X(\mathbb{R})$ when $X = G$ is a group variety, note first that this set coincides with the subgroup $\Gamma = G(\mathbb{Z}[1/p]) \subset G(\mathbb{R})$. 
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Consider therefore the group $G(\mathbb{R}) \times G(\mathbb{Q}_p)$, where the group $\Gamma = G(\mathbb{Z}[1/p])$ embeds diagonally as a lattice subgroup, namely as a discrete subgroup with finite covolume, (Borel-Harish Chandra 1960).
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We can then consider the finite-measure homogeneous space

$$Y = (G(\mathbb{R}) \times G(\mathbb{Q}_p))/G(\mathbb{Z}[1/p]).$$

on which $G(\mathbb{R}) \times G(\mathbb{Q}_p)$ acts (transitively) as a group of probability measure preserving transformations.
Counting lattice points

Consider the increasing sequence of height balls $B_h$ of $G(\mathbb{Q}_p)$

$$B_h = \{ g \in G(\mathbb{Q}_p) ; H_p(g) \leq h \}$$

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The Diophantine problem analogous to Schmidt’s theorem that we raised above is the problem of counting the number of rational points with $p$-height bounded by $h$ which are within $\delta$ of a point $x$ in $G(\mathbb{R})$. 
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- Clearly, this problem is identical to counting the number of points in the (diagonally embedded) lattice $\Gamma = G(\mathbb{Z}[1/p])$ which fall in the set

$$B(x, \delta) \times B_h \subset G(\mathbb{R}) \times G(\mathbb{Q}_p)$$
Effective mean ergodic theorems

We will now use the tools of ergodic theory, and solve the lattice point counting problem by analyzing the behavior of suitable averaging operators on the space $Y$. 
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Let $\beta_{\delta,h}$ denote the Haar-uniform probability measure supported on the sets $B(e,\delta) \times B_h \subset G(\mathbb{R}) \times G(\mathbb{Q}_p)$. Consider the operators

$$ \pi_Y(\beta_{\delta,h}) : L^2(Y) \to L^2(Y) $$

defined by averaging over these sets, namely

$$ \pi_Y(\beta_{\delta,h}) \phi((x,y)) = $$

$$ = \frac{1}{m(B(e,\delta))m(B_h)} \int_{(u,v) \in (B(e,\delta) \times B_h)} \phi((u^{-1}x, v^{-1}y)) \, du \, dv. $$
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- Then the following effective mean ergodic theorem holds:

$$
\left\| \pi_Y(\beta_{\delta,h})(\phi) - \int_Y \phi d\mu_Y \right\|_2 \leq C_\eta \cdot \text{vol}(B_h)^{-\frac{1}{q_p(G)}} + \eta \|\phi\|_2.
$$
The rate of convergence amounts to an operator norm estimate of the averaging operators acting in the automorphic representation, which can be deduced from the spectrum of the unitary representation of the group $G(\mathbb{Q}_p)$ in $L^2(Y)$.

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We note that in order to derive the optimal Diophantine exponent and threshold in Khinchin’s theorem using this method, optimality of the operator norm estimate in $L^2$ is crucial. This amounts to $q_p(G) = 2$, or equivalently, the representation of $G(\mathbb{Q}_p)$ being tempered.
Regularity of domains

- For a locally compact second countable group $G$, a lattice subgroup $\Gamma$, and a family of sets $B_t$ let us assume the following:

A) the validity of the effective mean ergodic theorem in the $G$-action on $L^2(G/\Gamma)$, in the form

$$\|\beta_t f - \int_{G/\Gamma} f \, dm\|_2 \leq C \eta m(B_t) - \theta + \eta \|f\|_2$$

B) the regularity condition for the sets $B_t$ given by ($t > t_0$, $0 < \varepsilon < \varepsilon_0$)

$$m(O_{\varepsilon} B_t O_{\varepsilon}) \leq (1 + c \varepsilon) m(\cap u, v \in O_{\varepsilon} u B_t v)$$

where $O_{\varepsilon}$ is a family of decreasing neighborhoods of $e \in G$ satisfying:

$$m(O_{\varepsilon}) \geq C \varepsilon^d.$$
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Solution of the lattice point counting problem

- **Theorem** (Gorodnik+N, 2008.) For any lcsc group $G$ and any lattice $\Gamma$, under conditions A and B, the lattice point counting problem in the domains $B_t$ has the solution

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This is a 2-parameter family, and we would like to let $h \to \infty$ and $\delta \to 0$ simultaneously. This calls for an extension of the theorem stated which establishes an effective solution of the lattice point counting problem in a variable family of domains.
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- We establish such an extension under suitable restrictions limiting the speed at which $\delta$ can converge to 0, as a function of $h^{-1}$. This concludes the outline of the proof of the uniform analog Schmidt’s theorem in this context.