Correlations and nilsequences

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Ordinary Correlations

In Ergodic Theory

A system (X, μ, T) is a probability space (X, μ) endowed with a transformation $T \colon X \to X$ measurable, invertible and preserving μ.

Notation.

We omit the σ-algebra. $T^n = T \circ \cdots \circ T$; $T^n x = T^n(x)$.
If f is a function on X, $T^n f = f \circ T^n$.

Definition.

The correlation of 2 functions $f_0, f_1 \in L^2(\mu)$ is the sequence $\phi = (\phi(n))_{n\in\mathbb{Z}}$ defined by

$$\phi(n) = \int_X f_0 \cdot T^n f_1 \, d\mu.$$

Notation. $\mathsf{e}(t) = \exp(2\pi i t)$.

Some definitions.

• A trigonometric polynomial on $\mathbb{Z}$ is a sequence $\phi = (\phi(n))_{n\in\mathbb{Z}}$ of the form

$$\phi(n) = \sum_{j=1}^{m} c_j \, \mathsf{e}(nt_j)$$

where $c_j \in \mathbb{C}$ and $t_j \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ for $j = 1, \ldots, m$.

• A sequence $\phi \in \ell^\infty(\mathbb{Z})$ is almost periodic if it is the uniform limit of trigonometric polynomials.

• A sequence $\phi \in \ell^\infty(\mathbb{Z})$ tends to 0 in density if

$$\lim_{N\to+\infty} \sup_{M\in\mathbb{Z}} \frac{1}{N} \sum_{n=M}^{M+N-1} |\phi(n)|^2 = 0.$$

Proposition.

Let $f_0, f_1 \in L^2(\mu)$. The correlation

$$\phi(n) = \int_X f_0 \cdot T^n f_1 \, d\mu$$

is the sum of an almost periodic sequence and a sequence tending to 0 in density.

Proof. By the Spectral Theorem, ϕ is the Fourier-Stieltjes transform $\widehat{\sigma}$ of a (complex) measure σ of finite total variation on $\mathbb{T} = \mathbb{Z}/\mathbb{Z}$. Let σ_d and σ_c be the discrete and continuous parts of σ. Then the sequence $\widehat{\sigma_d}$ is almost periodic and $\widehat{\sigma_c}$ tends to 0 in density. $\square$

Correlations of bounded sequences

The correlation of $f_0, f_1 \in \ell^\infty(\mathbb{Z})$ is defined by

$$\phi(n) = \lim_{N \to +\infty} \frac{1}{N} \sum_{x=0}^{N-1} f_0(x) f_1(x+n)$$

if the limits exists. Otherwise, pass to a subsequence.

Proposition.

The correlation ϕ of 2 elements of $\ell^\infty(\mathbb{Z})$ is the sum of an almost periodic sequence and a sequence tending to 0 in density.

Proof. We reduce to the preceding case by using Furstenberg Correspondence Principle. □

Correlations in the finite universe

Notation. $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$.

If A is a finite set, $|A| = \text{Card}(A)$. If f is a function on A,

$$\mathbb{E}_{x\in A}\, f(x) = \frac{1}{|A|}\sum_{x\in A} f(x).$$

Definition.

Let $N \in \mathbb{N}$. If f_0, f_1 are two functions on $\mathbb{Z}_N$, their correlation is defined by

$$\phi(n) = \mathbb{E}_{x\in\mathbb{Z}_N}\, f_0(x)\, f_1(x+n) \quad \text{for } n \in \mathbb{Z}_N.$$

Definition. A trigonometric polynomial on $\mathbb{Z}_N$ with m frequencies is a function $\phi\colon \mathbb{Z}_N \to \mathbb{C}$ of the form

$$\phi(n) = \sum_{j=1}^{m} c_j \, \mathrm{e}\Big(n \frac{\xi_j}{N}\Big)$$

where $c_j \in \mathbb{C}$ and $\xi_j \in \mathbb{Z}_N$, $j = 1, \ldots, m$.

Proposition.

For every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ with the following property:

For every $N \geq 1$ and all functions $f_0, f_1 \colon \mathbb{Z}_N \to \mathbb{C}$ with $|f_i| \leq 1$, the correlation

$$\phi(n) = \mathbb{E}_{x \in \mathbb{Z}_N} f_0(x) f_1(x+n)$$

is the sum

$$\phi(n) = \phi_{\text{struct}}(n) + \phi_{\text{sml}}(n)$$

of a trigonometric polynomial ϕ_{struct} with at most M frequencies, and a function ϕ_{sml} with $\|\phi_{\text{sml}}\|_2 \leq \varepsilon$.

Our problem:

Establish corresponding results for multiple correlations

In Ergodic Theory

Let (X, μ, T) be a system and $f_0, f_1, \ldots, f_k \in L^\infty(\mu)$. What can be said about the sequence $\phi = (\phi(n))_{n \in \mathbb{Z}}$ defined by

$$\phi(n) = \int f_0 \cdot T^n f_1 \cdot T^{2n} f_2 \ldots T^{kn} f_k \, d\mu \text{ ?}$$

More generally, let $(X, \mu, T_1, \ldots, T_k)$ be a system where the T_j's are commuting measure preserving transformations. For $f_0, f_1, \dots \in L^\infty(\mu)$, we also study the sequence ϕ given by

$$\phi(n) = \int f_0 \cdot T_1^n f_1 \cdot T_2^n f_2 \ldots T_k^n f_k \, d\mu.$$

We also consider sequences defined by several commuting transformations raised to polynomial powers, like

$$\phi(n) = \int f_0 \cdot T_1^n T_2^{n^2} f_1 \cdot T_1^{n^2+n} T_2^{n^2-n} f_2 \ldots d\mu.$$

Multiple correlations on $\mathbb{Z}_N$

Let $f_0, f_1, f_2, \ldots$ be functions on $\mathbb{Z}_N$. We want to describe correlation sequences of the form

$$\phi(n) = \mathbb{E}_{n\in\mathbb{Z}_N} f_0(x) \cdot f_1(x+n) \cdot f_2(x+2n) \ldots f_k(x+kn)$$

We also consider correlations arising from functions of 2 variables, like:

$$\phi(n) = \mathbb{E}_{x,y\in\mathbb{Z}_N} f_0(x,y) \cdot f_1(x+n,y) \cdot f_2(x,y+n) \cdot f_3(x+n,y+n)$$

and we also introduce polynomial terms, like

$$\mathbb{E}_{x,y\in\mathbb{Z}_N} f_0(x,y) \cdot f_1(x+n,y+n^2) \cdot f_2(x+n^2+n,y+n^2-n) \ldots$$

Most general case:

$$\phi(n) = \mathbb{E}_{x_1,\ldots,x_d\in\mathbb{Z}_N} \prod_{i=1}^{\ell} f_i\Big(x_1+p_{i,1}(n)\,,\, x_2+p_{i,2}(n)\,,\, \ldots, x_d+p_{i,d}(n)\Big)$$

where, for $i=1,\ldots,\ell$, f_i is a function on $\mathbb{Z}_N^d$ and $p_{i,j}\colon \mathbb{Z}\to\mathbb{Z}$ is a polynomial for $j=1,\ldots,d$.

The surprising fact is that in both cases the multiple correlation sequences are far from being arbitrary sequences.

The results in ergodic theory and in the finite universe look similar, and the general strategies of proofs are closely related.

However, the context and the tools are completely different.

Nilmanifolds, nilsequences

Definition.

Let $k \geq 1$, G a k-step nilpotent Lie group, connected and simply connected, and Γ a discrete co-compact subgroup of G. .

Then $X = G/\Gamma$ is a k-step nilmanifold.

The group G acts on X by left translations: $(g, x) \mapsto g \cdot x$

G is implicitly endowed with a right invariant Riemannian distance, and $X = G/\Gamma$ with the quotient distance.

1-step nilmanifolds are finite dimensional tori.

An example of a 2-step nilmanifold is the Heisenberg nilmanifold:

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \ ; \ \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \ ; X = G/\Gamma.$$

Definition.

Let $X = G/\Gamma$ be a k-step nilmanifold, $F\colon X \to \mathbb{C}$ a continuous function, $g \in G$ and $x \in X$. Then the sequence ϕ defined by

$$\Big(F(g^n \cdot x)\Big)_{n\in\mathbb{Z}}$$

is a k-step nilsequence.

Trigonometric polynomials are 1-step nilsequences.

If $p \in \mathbb{R}[t]$ is a polynomial of degree k then $(\mathrm{e}(p(n)))_{n\in\mathbb{Z}}$ is a k-step nilsequence.

But there are many others.

For the study of multiple correlations on $\mathbb{Z}_N$, we need a notion of complexity of a k-step nilsequences corresponding to the number of frequencies of a trigonometric polynomial.

For $M \in \mathbb{N}$, we do not give an explicit definition of a k-step nilsequences of complexity at most M.

For every k-step nilmanifold $X = G/\Gamma$ and every $C > 0$ there exists $M = M(X, C)$ such that every nilsequence ϕ of the form

$$\phi(n) = F(g^n \cdot x) \text{ where } g \in G,\ x \in X,\ F \colon X \to \mathbb{C} \text{ and } \|F\|_{\mathrm{Lip}(X)} \leq C$$

has complexity $\leq M$.

Conversely, the family of k-step nilsequences of complexity $\leq M$ can be defined in this way.

Results in Ergodic Theory

Theorem [Bergelson, H. & Kra, 2005]

Let $k \geq 1$, (X, μ, T) an ergodic system and $f_0, f_1, \ldots, f_k \in L^\infty(\mu)$. Then the correlation ϕ defined by

$$\phi(n) = \int f_0 \cdot T^n f_1 \cdot T^{2n} f_2 \ldots, T^{kn} f_k \, d\mu$$

can be decomposed as a sum

$$\phi = \phi_{\text{struct}} + \phi_{\text{null}}$$

where ϕ_{struct} is the uniform limit of k-step nilsequences and ϕ_{null} tends to 0 in density.

Generalization to non ergodic systems by using a result of Leibman (2015): every integral of a family of nilsequences is the uniform limit of nilsequences.

The original method relies on a Structure Theorem and can not be used to study correlations arising from a system with several commuting transformations: no known structure theorem.

A similar result holds for all correlations, but with a completely different proof.

Theorem [Frantzikinakis 2015]

All multiple correlations in Ergodic Theory can be decomposed as a sum of a nilsequence and a small error term.

Here "small" refers to the seminorm

$$\|\phi\|_2 = \Big(\lim_{N\to+\infty} \sup_{M\in\mathbb{Z}} \frac{1}{N} \sum_{n=M}^{M+N-1} |\phi(n)|^2 \Big)^{1/2}$$

on $\ell^\infty(\mathbb{Z})$.

Multiple Correlations in the Finite Universe

Theorem.

For every k and every $\varepsilon > 0$ there exists M with the following property. For every N and all functions $f_0, f_1, \ldots, f_k \colon \mathbb{Z}_N \to \mathbb{C}$ with $|f_i| \leq 1$, the correlation

$$\phi(n) = \mathbb{E}_{x \in \mathbb{Z}_N} f_0(x) \cdot f_1(x+n) \ldots f_k(x+kn)$$

can be written as a sum

$$\phi(n) = \phi_{\mathsf{nil}}(n) + \phi_{\mathsf{sml}}(n)$$

where ϕ_{nil} is a k-step nilsequence of complexity $\leq M$ and $\|\phi_{\mathsf{sml}}\|_2 < \varepsilon$.

The basic tool is the Inverse Theorem for Gowers Norms of Green, Tao & Ziegler (2012).

There is no Inverse Theorem for the "box norms" and the next result needs another method.

Theorem

For every k and every $\varepsilon > 0$ there exists M with the following property:

For every N and all functions $f_0, f_1, \ldots, f_k : \mathbb{Z}_N^k \to \mathbb{C}$ with $|f_i| \leq 1$, the correlation

$$\phi(n) = \mathbb{E}_{x_1,\ldots,x_k \in \mathbb{Z}_N} f_0(x_1, x_2, \ldots, x_k) \cdot f_1(x_1 + n, x_2, \ldots, x_k) \cdot \\ \cdot f_2(x_1, x_2 + n, \ldots, x_k) \ldots f_k(x_1, x_2, \ldots, x_k + n)$$

can be written as a sum

$$\phi(n) = \phi_{\mathrm{nil}}(n) + \phi_{\mathrm{sml}}(n)$$

where ϕ_{nil} is a k-step nilsequence of complexity $\leq M$ and $\|\phi_{\mathrm{sml}}\|_2 < \varepsilon$.

We have a similar result for all polynomial multiple correlations.

Most general form of the Theorem

Let $d, \ell \in \mathbb{N}$ and let $p_{i,j} \colon \mathbb{Z} \to \mathbb{Z}$, $1 \leq i \leq \ell$, $1 \leq j \leq d$ be polynomials. Then there exists $k \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists M with the following property.

For every N, for all functions $f_1, \ldots, f_\ell \colon \mathbb{Z}_N \to \mathbb{C}$ with $|f_j| \leq 1$, the correlation ϕ given by

$$\phi(n) = \mathbb{E}_{x_1,\ldots,x_d \in \mathbb{Z}_N} \prod_{i=1}^{\ell} f_i\Big(x_1 + p_{i,1}(n)\,,\, x_2 + p_{i,2}(n)\,,\, \ldots, x_d + p_{i,d}(n)\Big)$$

can be decomposed as a sum

$$\phi(n) = \phi_{\mathsf{nil}}(n) + \phi_{\mathsf{sml}}(n)$$

where

ϕ_{nil} is a k-step nilsequence of complexity $\leq M$ and $\|\phi_{\mathsf{sml}}\|_2 < \varepsilon$.

Tools for the proof of the theorem in the finite case

Gowers norms on $\mathbb{Z}_N$

Definition.

Let f be a function on $\mathbb{Z}_N$.

$$\|f\|_{U^1} = \left|\mathbb{E}_{x\in\mathbb{Z}_N} f(x)\right|.$$

For $k \geq 1$, writing $f_h(x) = f(x+h)$,

$$\|f\|_{U^{k+1}} = \left(\mathbb{E}_{h\in\mathbb{Z}_N} \| f \cdot \overline{f_h} \|_{U^k}^{2^k} \right)^{1/2^{k+1}}.$$

For $k \geq 2$, $f \mapsto \|f\|_{U^k}$ is a norm.

A direct computation gives $\|f\|_{U^2} = \| \widehat{f} \|_{\ell^4(\mathbb{Z}_N)}$ but there is no similar formula for $\|f\|_{U^k}$, $k > 2$.

$$\|f\|_{U^3}^8 = \mathbb{E}_{x,h_1,h_2,h_3\in\mathbb{Z}_N}\Big(f(x)\overline{f}(x+h_1)\overline{f}(x+h_2)f(x+h_1+h_2) \overline{f}(x+h_3)f(x+h_1+h_3)f(x+h_2+h_3)\overline{f}(x+h_1+h_2+h_3) \Big)$$

Arithmetic Regularity Lemma

Theorem. [Green & Tao, 2010]

Let $k \geq 1$ and $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ a function.
Then for every $\delta > 0$ there exists M with the following property. For every N, every function $f\colon \mathbb{Z}_N \to \mathbb{R}$ with $|f| \leq 1$ admits a decomposition

$$f(n) = f_{\mathsf{nil}}(n) + f_{\mathsf{sml}}(n) + f_{\mathsf{unif}}(n)$$

where

- f_{nil} is a k-step nilsequence of complexity $\leq M$;
- $\|f_{\mathsf{sml}}\|_2 \leq \delta$;
- $\|f_{\mathsf{unif}}\|_{U^{k+1}} \leq \Phi(M)^{-1}$.

Moreover, $|f_{\mathsf{nil}}| \leq 1$ and $|f_{\mathsf{nil}} + f_{\mathsf{sml}}| \leq 1$.

The proof used the Inverse Theorem for the Gowers Norms of Green, Tao and Ziegler (2012).

Sketch of the proof in a particular case

Theorem

For every k and every $\varepsilon > 0$ there exists with the following property. For every N and all function $f_0, f_1, \ldots, f_k : \mathbb{Z}_N^k \to \mathbb{C}$ with $|f_i| \leq 1$, the correlation

$$\phi(n) = \mathbb{E}_{x_1,\ldots,x_k \in \mathbb{Z}_N} f_0(x_1, x_2, \ldots, x_k) \cdot f_1(x_1 + n, x_2, \ldots, x_k) \cdot \\ \cdot f_2(x_1, x_2 + n, \ldots, x_k) \ldots f_k(x_1, x_2, \ldots, x_k + n)$$

can be written as a sum

$$\phi(n) = \phi_{\text{nil}}(n) + \phi_{\text{sml}}(n)$$

where ϕ_{nil} is a k-step nilsequence of complexity $\leq M$ and $\|\phi_{\text{sml}}\|_2 < \varepsilon$.

Proof.

The idea is to use the Regularity Lemma to obtain a decomposition of ϕ, and not of the functions f_i.

$$\phi(n) = \mathbb{E}_{x_1,\dots,x_k \in \mathbb{Z}_N} f_0(x_1, x_2, \dots, x_k) \cdot f_1(x_1 + n, x_2, \dots, x_k) \cdot$$
$$\cdot f_2(x_1, x_2 + n, \dots, x_k) \dots f_k(x_1, x_2, \dots, x_k + n)$$

Let $\Phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function and $\delta > 0$ a constant to be specified later. Using the Regularity Lemma, we write

$$\phi = \phi_{\text{nil}} + \phi_{\text{sml}} + \phi_{\text{unif}}$$

where

- ϕ_{nil} is a k-step nilsequence of complexity $\leq M$;
- $\|\phi_{\text{sml}}\|_2 \leq \delta$;
- $\|\phi_{\text{unif}}\|_{U^{k+1}} \leq \Phi(M)^{-1}$

and moreover $|\phi_{\text{nil}}| \leq 1$ and $|\phi_{\text{nil}} + \phi_{\text{sml}}| \leq 1$.

We show that in fact we have $\|\phi_{\text{unif}}\|_2 \leq \varepsilon/2$

We write

$$\begin{aligned}\|\phi_{\text{unif}}\|_2^2 &= \mathbb{E}_{n\in\mathbb{Z}_N} |\phi_{\text{unif}}(n)|^2 \\ &= \underbrace{\mathbb{E}_{n\in\mathbb{Z}_N} \phi_{\text{unif}}(n)\phi(n)}_{\text{first term}} - \underbrace{\mathbb{E}_{n\in\mathbb{Z}_N} \phi_{\text{unif}}(n)\phi_{\text{sml}}(n)}_{\text{second term}} - \underbrace{\mathbb{E}_{n\in\mathbb{Z}_N} \phi_{\text{unif}}(n)\phi_{\text{nil}}(n)}_{\text{third term}}\end{aligned}$$

and we bound each term on the right hand side.

First term.

Proposition.

Let $f_0, \ldots, f_k \colon \mathbb{Z}_N \to \mathbb{C}$ with $|f_i| \le 1$ and let $\psi \colon \mathbb{Z}_N \to \mathbb{C}$. Then

$$\Big|\mathbb{E}_{n\in\mathbb{Z}_N} \psi(n)\, \mathbb{E}_{x_1,\ldots,x_k\in\mathbb{Z}_N} f_0(x_1, x_2, \ldots, x_k) \cdot f_1(x_1 + n, x_2, \ldots, x_k) \cdot \\ \cdot f_2(x_1, x_2 + n, \ldots, x_k) \ldots f_k(x_1, x_2, \ldots, x_k + n)\Big| \le c\,\|\psi\|_{U^k}.$$

Therefore we have

$$\left|\mathbb{E}_{n\in\mathbb{Z}_N} \phi_{\text{unif}}(n)\, \phi(n)\right| \le c\,\Phi(M)^{-1} \le \varepsilon^2/12.$$

Second term

Since $\|\phi_{\text{unif}}\|_\infty \le 2$ and $\|\phi_{\text{sml}}\|_2 \le \delta$,

$$\left| \mathbb{E}_{n\in\mathbb{Z}_N}\, \phi_{\text{unif}}(n)\, \phi_{\text{sml}}(n) \right| \le 2\delta < \varepsilon^2/12.$$

Third term

Theorem.

Let $k \ge 1$. For every M and every $\delta > 0$ there exists $C(M,\delta) > 0$ such that for every function $\theta\colon \mathbb{Z}_N \to \mathbb{C}$ and every k-step nilsequence ψ of complexity $\le M$, we have

$$\left| \mathbb{E}_{n\in\mathbb{Z}_N}\, \theta(n)\, \psi(n) \right| \le C(M,\delta)\, \|\theta\|_{U^{k+1}} + \delta\, \|\theta\|_\infty.$$

Therefore we have

$$\left| \mathbb{E}_{n\in\mathbb{Z}_N} \phi_{\text{unif}}(n) \phi_{\text{nil}}(n) \right| \le C(M,\delta)\Phi(M)^{-1} + \delta < \varepsilon^2/12.$$

Putting the bounds together we obtain $\|\phi_{\mathsf{unif}}\|_2 \leq \varepsilon/2$.

We have

$$\phi = \phi_{\mathsf{nil}} + (\phi_{\mathsf{sml}} + \phi_{\mathsf{unif}})$$

and $\|\phi_{\mathsf{sml}} + \phi_{\mathsf{unif}}\|_2 < \varepsilon$. $\square$

Thank you for your attention