Ergodicity of the Liouville system implies the Chowla conjecture

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Luminy, December 2016

The Liouville function

- **Liouville function:** If $n = p_1^{a_1} \cdots p_k^{a_k}$, then $\lambda(n) = (-1)^{a_1+\cdots+a_k}$. Its sign is expected to be "randomly distributed" on the integers.

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- But... Not even known that all size 4 patterns occur infinitely often!

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Sign patterns in the range of λ

Some known results about sign patterns taken by λ :

- Size 1 patterns: Both occur with density $\frac{1}{2}$ (PNT).
- Size 2 patterns: All four occur infinitely often (Harman, Pintz, Wolke 85), positive lower density (Matomäki, Radziwiłł, 2015), logarithmic density $\frac{1}{4}$ (Tao 2015).
- Size 3 patterns: All eight occur infinitely often (Hilderbrand 1986), positive lower density (Matomäki, Radziwiłł, Tao, 2015).
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Chowla conjecture

Chowla Conjecture (1965)

If $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ are distinct, then

$$\lim_{M\to\infty} \mathbb{E}_{m\in[M]} \lambda(m+n_1)\cdots\lambda(m+n_\ell) = 0.$$

- $\ell = 1$ (PNT): $\mathbb{E}_{m\in\mathbb{N}}\lambda(m) = 0$.
- $\ell = 2$ (Tao 2015): Proof for logarithmic averages. For every $n \in \mathbb{N}$

$$\lim_{M\to\infty} \frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \lambda(m)\,\lambda(m+n) = 0.$$

- Open for $\ell = 3$ even for logarithmic averages for all choices of distinct $n_1, \ldots, n_\ell \in \mathbb{N}$.
- Averaged version (Matomäki, Radziwiłł, Tao 2015):

$$\lim_{N\to\infty} \mathbb{E}_{n_1,\ldots,n_\ell\in[N]} \limsup_{M\to\infty} \left|\mathbb{E}_{m\in[M]}\lambda(m+n_1)\cdots\lambda(m+n_\ell)\right| = 0.$$

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A simplifying assumption

For clarity purposes and in order to ease notation we assume

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The Liouville function **admits correlations,** meaning, the limit

$$\lim_{M\to\infty} \mathbb{E}_{m\in[M]}\lambda(m+n_1)\cdots\lambda(m+n_\ell)$$

exists for every $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$.

In the general case, we work with any subsequence of intervals $([M_k])_{k\in\mathbb{N}}$ along which λ **admits correlations.** Then we get Chowla-type results for **logarithmic averages** along $([M_k])_{k\in\mathbb{N}}$.

Notation

- $\mathbb{E}_{n\in\mathbb{N}} a(n) = \lim_{N\to\infty} \mathbb{E}_{n\in[N]} a(n)$ if the limit exists.
- $\bar{\mathbb{E}}_{n\in\mathbb{N}} a(n) = \limsup_{N\to\infty} \mathbb{E}_{n\in[N]} a(n)$ if $a(n) \geq 0$.

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Ergodic reinterpretation of Chowla conjecture

Furstenberg Correspondence Principle

If $a \in \ell^\infty(\mathbb{N})$ admits correlations, then there exist a measure preserving system $(X, \mathcal{X}, \mu, T)$ and a function $f \in L^\infty(\mu)$ such that

$$\int T^{n_1} f \cdots T^{n_\ell} f \, d\mu = \mathbb{E}_{m \in \mathbb{N}} a(m + n_1) \cdots a(m + n_\ell)$$

for every $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}$. If $a = \lambda$, we call the system the **Liouville system** and denote the corresponding measure by μ_λ.

- $X = D^{\mathbb{Z}}$, $(Tx)(k) = x(k+1)$, $f(x) = x(0)$, only μ varies.
- Chowla conjecture $\Rightarrow$ Liouville system is a Bernoulli system.
- Main goal: ergodicity of the Liouville system $\Rightarrow$ Chowla conjecture.
- Ergodic point of view also used (for example) by
  - Sarnak to study properties of the Möbius system and
  - el Abdalaoui, Kułaga-Przymus, Lemańczyk, de la Rue, to study the Chowla and the Sarnak conjecture.

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Some facts about the Liouville system

- (Matomäki, Radziwiłł 2015): f **is orthogonal to the invariant factor** of the Liouville system since

$$\mathbb{E}_{n\in\mathbb{N}} \int f \cdot T^n f \, d\mu_\lambda = 0 \iff \lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} |\mathbb{E}_{n\in[m,m+N]} \lambda(n)| = 0.$$

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- **It is not known if f is orthogonal to $\mathcal{Z}_1(\mu_\lambda)$.** If $\mu_\lambda = \int \mu_x \, d\mu_\lambda$ is the ergodic decomposition of μ_λ, then $\mathbb{E}_{n\in\mathbb{N}} \int | \int f \cdot T^n f \, d\mu_x | \, d\mu_\lambda = 0 \iff$

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and this is equivalent to $\|\lambda\|_{U^2(\mathbb{N})} = 0$ (to be defined shortly).

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Ergodicity implies the Chowla conjecture

Main Result (assumes λ admits correlations)

If the Liouville system is ergodic, then the Chowla conjecture holds.

Equivalently, if the Liouville function is generic for an ergodic measure, then the Chowla conjecture holds.

Main Result (no implicit assumption)

If the Liouville function admits correlations for logarithmic averages along $([M_k])$ and the corresponding system is ergodic, then the Chowla (and Sarnak) conjecture hold for **logarithmic averages along** $([M_k])$.

Averaging operation used: $\frac{1}{\log M_k} \sum_{m \in [M_k]} \frac{1}{m} \cdots$.

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Main steps in the proof

The proof contains three main ingredients:

1. Tao (2015): Local uniformity of the Liouville function implies the Chowla conjecture (for logarithmic averages if existence of correlations is not assumed).
2. An inverse theorem for local uniformity seminorms of ergodic sequences.
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Local uniformity seminorms

Definition (Host, Kra 2009)

If $a \in \ell^\infty(\mathbb{N})$ admits correlations, we let $(S_r a)(n) := a(n+r)$ and

$$\|a\|_{U^1(\mathbb{N})}^2 := \mathbb{E}_{r\in\mathbb{N}}\mathbb{E}_{n\in\mathbb{N}} a(n+r)\cdot\overline{a(n)}, \quad \|a\|_{U^{s+1}(\mathbb{N})}^{2^{s+1}} := \mathbb{E}_{r\in\mathbb{N}} \|S_r a \cdot \overline{a}\|_{U^s(\mathbb{N})}^{2^s}.$$

- All limits can be shown to exist (using the ergodic reinterpretation).
- $\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r,s\in\mathbb{N}}\big(\mathbb{E}_{n\in\mathbb{N}}\, a(n)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n+r+s)\big).$
- If $(a(n))_{n\in\mathbb{N}}$ is ergodic, then $\|a\|_{U^1(\mathbb{N})} = \big|\mathbb{E}_{n\in\mathbb{N}} a(n)\big|$ and

$$\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r\in\mathbb{N}}\big|\mathbb{E}_{n\in\mathbb{N}} a(n+r)\cdot\overline{a(n)}\big|^2.$$

- Ergodic reinterpretation: If $(X,\mathcal{X},\mu,T)$ is the system and f is the function associated to $(a(n))_{n\in\mathbb{N}}$, then $\|a\|_{U^s(\mathbb{N})} = \|\!|f|\!\|_s$ where $\|\!|\cdot|\!\|_s$ are the Host-Kra seminorms:

$$\|\!|f|\!\|_1^2 = \mathbb{E}_{r\in\mathbb{N}}\int T^r f\cdot\overline{f}\,d\mu, \quad \|\!|f|\!\|_{s+1}^{2^{s+1}} := \mathbb{E}_{r\in\mathbb{N}}\|\!|T^r f\cdot\overline{f}|\!\|_s^{2^s}.$$

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$$\|a\|_{U^1(\mathbb{N})}^2 := \mathbb{E}_{r\in\mathbb{N}}\mathbb{E}_{n\in\mathbb{N}} a(n+r)\cdot\overline{a(n)}, \quad \|a\|_{U^{s+1}(\mathbb{N})}^{2^{s+1}} := \mathbb{E}_{r\in\mathbb{N}} \|S_r a\cdot \overline{a}\|_{U^s(\mathbb{N})}^{2^s}.$$

- All limits can be shown to exist (using the ergodic reinterpretation).
- $\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r,s\in\mathbb{N}}\big(\mathbb{E}_{n\in\mathbb{N}}\, a(n)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n+r+s)\big).$
- If $(a(n))_{n\in\mathbb{N}}$ is ergodic, then $\|a\|_{U^1(\mathbb{N})} = \left|\mathbb{E}_{n\in\mathbb{N}} a(n)\right|$ and

$$\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r\in\mathbb{N}}\left|\mathbb{E}_{n\in\mathbb{N}} a(n+r)\cdot\overline{a(n)}\right|^2.$$

- Ergodic reinterpretation: If $(X, \mathcal{X}, \mu, T)$ is the system and f is the function associated to $(a(n))_{n\in\mathbb{N}}$, then $\|a\|_{U^s(\mathbb{N})} = |||f|||_s$ where $|||\cdot|||_s$ are the Host-Kra seminorms:

$$|||f|||_1^2 = \mathbb{E}_{r\in\mathbb{N}}\int T^r f\cdot\overline{f}\,d\mu, \quad |||f|||_{s+1}^{2^{s+1}} := \mathbb{E}_{r\in\mathbb{N}} |||T^r f\cdot\overline{f}|||_s^{2^s}.$$

Local uniformity implies Chowla

Theorem (Tao 2015)

$\|\lambda\|_{U^s(\mathbb{N})} = 0$ *for every* $s \in \mathbb{N} \iff$ *The Chowla conjecture is satisfied.*

- Gowers uniformity is known for λ (Green, Tao, Ziegler 2012), but this is a much weaker condition than local uniformity.
- $\|\lambda\|_{U^1(\mathbb{N})} = 0 \iff \lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} |\mathbb{E}_{n\in[m,m+N]}\lambda(n)| = 0$ which is known by Matomäki, Radziwiłł (2015).
- $\|\lambda\|_{U^2(\mathbb{N})} = 0$ is an open problem. It is equivalent to

$$\lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} \sup_t |\mathbb{E}_{n\in[m,m+N]}\lambda(n)\, e(nt)| = 0.$$

Hence, our main result follows from:

Theorem ($U^s(\mathbb{N})$-uniformity for λ)

If the Liouville system is ergodic, then $\|\lambda\|_{U^s(\mathbb{N})} = 0$ *for every* $s \in \mathbb{N}$.

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Step 1: An inverse theorem for ergodic sequences

Definition (Nilsequences)

- (Bergelson, Host, Kra 05) $X = G/\Gamma$ is an s-step nilmanifold, $b \in G$, $\Psi \in C(X)$, then $\psi(n) = \Psi(b^n \cdot e_X)$ is an ***s*-step nilsequence.**
- (Nilsequences of **bdd complexity on X**) If $X = G/\Gamma$, we let

$$\Psi_X := \{(\Psi(b^n \cdot e_X))_{n\in\mathbb{N}},\ b \in G,\ \|\Psi\|_{\mathrm{Lip}(X)} \leq 1\}.$$

Theorem (Inverse theorem for $U^s(\mathbb{N})$-seminorms)

Let $a \in \ell^\infty(\mathbb{N})$ be an ergodic sequence. Then $\|a\|_{U^{s+1}(\mathbb{N})} = 0$ if and only if for every s-step nilsequence ϕ and every $(s-1)$-step nilmanifold Y

$$\lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} \sup_{\psi\in\Psi_Y} |\mathbb{E}_{n\in[m,m+N]} a(n)\,\phi(n)\,\psi(n)| = 0.$$

For $s = 1$ the inverse condition is satisfied if for every $t \in \mathbb{R}$ (no sup!)

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Inverse theorem (Proof for $s = 2$)

Suppose that $\|a\|_{U^2(\mathbb{N})} > 0$.

1. **Ergodicity** implies $\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r\in\mathbb{N}}\big|\mathbb{E}_{n\in\mathbb{N}}a(n)\cdot\overline{a(n+r)}\big|^2$, hence
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where $A(r) := \mathbb{E}_{n\in\mathbb{N}}\overline{a(n+r)}\cdot a(n),\ r\in\mathbb{N}$.
2. It is not hard to show that
$$A(r) = \sum_{k=1}^{\infty} c_k e(r\alpha_k) + E(r),$$
where $\sum_{k=1}^{\infty}|c_k| < \infty$, $\mathbb{E}_{r\in\mathbb{N}}|E(r)| = 0$. Hence, for some $\alpha\in\mathbb{R}$
$$\limsup_{R\to\infty}\big|\mathbb{E}_{r\in[R]}\big(\mathbb{E}_{n\in\mathbb{N}}a(n+r)\cdot\overline{a(n)}\cdot e(r\alpha)\big)\big| > 0,$$
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Inverse theorem (Sketch of proof for $s = 3$)

Suppose that $\|a\|_{U^3(\mathbb{N})} > 0$.

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$$A(r,s) := \mathbb{E}_{n\in\mathbb{N}} a(n+r+s)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n).$$

2. Using ergodic theory (a structure theorem of Host and Kra (05)) we get

$$A(r,s) = \Phi(r,s) + E(r,s),$$

such that

- $\Phi(r,s) = \mathbb{E}_{n\in\mathbb{N}}\, \phi(n+r+s)\,\overline{\phi(n+r)}\,\overline{\phi(n+s)}\,\phi(n)$ where ϕ **is a** 2 **step nilsequence;**
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3. For convenience say $\Phi(r,s) = e(rs\alpha)$. Then

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4. We deduce from the previous step that

$$\|b\|_{U^2(\mathbb{N})} > 0.$$

Using a finitistic decomposition result of Green and Tao we get

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Theorem (Orthogonality of λ with nilsequences)

Suppose that the Liouville system is ergodic. Then for every $s \in \mathbb{N}$, for every s-step nilsequence ϕ and every $(s-1)$-step nilmanifold Y

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- Flaminio, Fraczek, Kułaga-Przymus, Lemańczyk (2016): Variant without the sup.
- Proof by induction on $s \in \mathbb{N}$. Schematically

MRT15 $\Rightarrow$ $\lambda \perp$ 1-step nil ($s = 1$ case)

$\Rightarrow^{\text{inv thm}}$ $\lambda \perp$ sup(1-step nil)

$\Rightarrow^{\text{dyn arg}}$ $\lambda \perp$ 2-step nil + sup(1-step nil) ($s = 2$ case)

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The inverse condition for the Liouville function

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\begin{aligned}
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\Rightarrow^{\text{dyn arg}}\ & \lambda \perp \text{2-step nil} + \sup(\text{1-step nil}) && (\mathbf{s}=\mathbf{2}\text{ case})\\
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\Rightarrow^{\text{inv thm}} \quad & \lambda \perp \sup(\text{2-step nil}) && \\
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\end{aligned}$$

The inverse condition for the Liouville (induction step)

- Suppose statement holds for $(s-1)$. Want to show: If $X = G/\Gamma$ is an s-step nilmanifold, $b \in G$, $\Phi \in C(X)$, and Y is an $(s-1)$-step nilmanifold, then

$$\lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} \sup_{\psi\in\Psi_Y} |\mathbb{E}_{n\in[m,m+N]}\lambda(n)\,\Phi(b^n\cdot e_X)\,\psi(n)| = 0.$$

- We can assume that Φ **is a vertical nilcharacter,** meaning, if $K_s = G_s/(G_s\cap\Gamma)$, then for some $\chi \in \widehat{K_s}$

$$\Phi(u\cdot x) = \chi(u)\,\Phi(x), \quad \text{for every } u \in G_s.$$

- If χ is trivial, then Φ factors through an $(s-1)$-step nilmanifold $\Rightarrow$

  $\Phi(b^n\cdot e_X)$ is an $(s-1)$-step nilsequence.

  Induction hypothesis and inverse theorem $\Rightarrow \|\lambda\|_{U^s(\mathbb{N})} = 0 \Rightarrow$

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  for every $(s-1)$-step nilmanifold Y (use of van der Corput lemma).

- So we can assume that χ **is non-trivial.**

The inverse condition for the Liouville (induction step)

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Reduction to a dynamical property

Using an orthogonality criterion of Kátai (86) we reduce matters even further to showing the following statement of purely dynamical context:

Theorem (Orthogonality of irrational nilsequences)

Let $X = G/\Gamma$ be a connected s-step nilmanifold, $b \in G$ ergodic, Φ be a non-trivial nilcharacter of X, Y be an $(s-1)$-step nilmanifold, $p, q \in \mathbb{N}$ with $p \neq q$, $(I_N)_{N\in\mathbb{N}}$ intervals with $|I_N| \to \infty$. Then

$$\lim_{N\to\infty} \sup_{\psi\in\Psi_Y} \left|\mathbb{E}_{n\in I_N} \Phi(b^{pn}\cdot e_X)\,\overline{\Phi(b^{qn}\cdot e_X)}\,\psi(n)\right| = 0.$$

- Model case: $\Phi(b^n \cdot e_X) = e(n^s\beta)$ with β irrational. Need to show

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Sketch of proof of the dynamical property

- Apply van der Corput $s-1$ times reduces matters to showing: $\|\Phi \otimes \overline{\Phi}\|_{s,Y} = 0$, for the rotation by (b^p, b^q) acting on

$$Y = \overline{\{(b^{pn} \cdot e_X, b^{qn} \cdot e_X), n \in \mathbb{N}\}}.$$

We know that Y is a nilmanifold (by Lesigne 91 and Leibman 05).

- Key observation: $Y = H/\Delta$ where $\Gamma \times \Gamma \subset H$ and

$$(u^{p^s}, u^{q^s}) \in H_s \quad \text{for every } u \in G_s.$$

Why? We get $(g^p, g^q) \in H \cdot (G_2 \times G_2)$ because b is totally ergodic on X. Then take iterative commutators $s-1$ times.

- It follows that $\chi \otimes \overline{\chi}$ is non-trivial on H_s, hence $\Phi \otimes \overline{\Phi}$ is a nontrivial nilcharacter of the s-step nilmanifold Y.
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Open problems

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Suppose that the Liouville function admits correlations. Show that the induced system **is ergodic.**

- A variant for logarithmic averages would imply the Chowla (and Sarnak) conjecture for logarithmic averages.

Easier Problem

Suppose that the Liouville function admits correlations. Show that the induced system **is not a (non-ergodic) mixture of circle rotations.**

- It is not clear how to exclude the possibility that λ has the same statistics with a sequence consisting of 1-step nilsequences of bdd complexity on larger and larger blocks that exhaust the integers.

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