Normal Subsequences of Automatic Sequences

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Thue-Morse sequence $(t_n)_{n \geq 0}$:
★ Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n\geq 0}\):

\[0\]
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

01
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

0110
Thue-Morse sequence $\{t_n\}_{n \geq 0}$: 

$01101001$
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

0110100110010110
Thue-Morse sequence $(t_n)_{n \geq 0}$:

01101001100101101001011001101001
Thue-Morse sequence $(t_n)_{n \geq 0}$:

$$011010011001011010010110011010011001011001101 \cdots$$

$$t_0 = 0, \quad t_{2^n + k} = 1 - t_k \quad (0 \leq k < 2^n)$$
Thue-Morse sequence $(t_n)_{n \geq 0}$:

\[ 011010011001011010010110011010011001011001101 \ldots \]

\[ t_0 = 0, \quad t_{2^n+k} = 1 - t_k \quad (0 \leq k < 2^n) \]

\[ t_n = s_2(n) \mod 2 \]

\[ n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \quad \varepsilon_i(n) \in \{0, 1, \ldots, q - 1\}, \quad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n) \]
Thue-Morse sequence $(t_n)_{n \geq 0}$:

\[ 01101001100101101001011001101001100101101 \cdots \]

\[ t_0 = 0, \quad t_{2^n+k} = 1-t_k \quad (0 \leq k < 2^n) \quad \text{or} \quad t_{2k} = t_k, \ t_{2k+1} = 1-t_k \]

\[ t_n = s_2(n) \mod 2 \]

\[
\begin{align*}
    n &= \sum_{i=0}^{\ell-1} \varepsilon_i(n)q^i \\
    \varepsilon_i(n) &\in \{0, 1, \ldots, q-1\}, \quad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)
\end{align*}
\]
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}:\)

\[011010011001011010010110011010011001011001101\cdots\]
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\[01101001100101101001011001101001101001011001101 \cdots\]
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Thue-Morse sequence $(t_n)_{n \geq 0}$:

011010011001011010010110011010011001011001101101101 \ldots
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
011010011001011010010110011010011001011001101 \ldots
\]

\[
\# \{0 \leq n < N : t_n = 0\} \sim \frac{N}{2}
\]
★ Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}:(011010011001011010010110011010011001011001101\ldots)

\[\# \{0 \leq n < N : t_n = 0\} \sim \frac{N}{2}\]

The letters 0 and 1 appear with asymptotic frequency \(\frac{1}{2}\).
Thue-Morse sequence

- TM sequence is **not periodic** and **cubeless**.
- TM sequence is **almost periodic**:
  Every appearing consecutive block appears infinitely many times with bounded gaps.

- **Subword complexity is linear**: \( p_k \leq \frac{10}{3} k \)
  \( p_k \) ... subword complexity (number of different consecutive blocks of length \( k \) that appear in the TM sequence).

- **Zero topological entropy** of the corresponding dynamical system:
  \[
  h = \lim_{k \to \infty} \frac{1}{k} \log p_k = 0
  \]

- **Linear subsequences** \((t_{an+b})_{n \geq 0}\) have the same properties.
- The TM sequence and its linear subsequences are **automatic sequences**.
Thue-Morse sequence

Automaton that generates the Thue-Morse sequence:

\[ t_n = \sum_{j \geq 0} \varepsilon_j(n) \mod 2 \]
Rudin-Shapiro sequence

Rudin-Shapiro sequence \((r_n)_{n \geq 0}\):
Rudin-Shapiro sequence (\(r_n\))_{n \geq 0}:

00010010000111010010111000100001001000011101111 \ldots
Rudin-Shapiro sequence

Rudin-Shapiro sequence \((r_n)_{n \geq 0}\):

\[
0001001000011101000100101110001000011101111 \ldots
\]

\[
r_0 = 0, \quad r_{2k} = r_k, \quad r_{2k+1} = \begin{cases} r_k & \text{if } k \text{ is even}, \\ 1 - r_k & \text{if } k \text{ is odd}. \end{cases}
\]
Rudin-Shapiro sequence

Rudin-Shapiro sequence \((r_n)_{n \geq 0}\):

\[
000100100001110100010010111000100001001000011101111 \ldots
\]

\[
r_0 = 0, \quad r_{2k} = r_k, \quad r_{2k+1} = \begin{cases} r_k & \text{if } k \text{ is even}, \\ 1 - r_k & \text{if } k \text{ is odd}. \end{cases}
\]

\[
r_n = \sum_{i \geq 0} \varepsilon_i(n)\varepsilon_{i+1}(n) \mod 2
\]

\[
n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i, \quad \varepsilon_i(n) \in \{0, 1, \ldots, q - 1\}
\]
Rudin-Shapiro sequence

Automaton that generates the Rudin-Shapiro sequence:

\[ r_n = \sum_{j \geq 0} \varepsilon_j(n)\varepsilon_{j+1}(n) \mod 2 \]
Automatic sequences

Definition
A sequence \((u_n)_{n \geq 0}\) is called a \(q\)-automatic sequence, if \(u_n\) is the output of an automaton when the input is the \(q\)-ary expansion of \(n\).

\[
\begin{align*}
(u_n)_{n \geq 0} : & \quad \text{aaaaabaabaabaaabbaaabaaabbaaabbaaabbaabaaabbaaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabbaab...}
\end{align*}
\]
Automatic sequences

- **Sum-of-digits-function**: $u_n = s_q(n) \mod m$

- **q-additive function** modulo $m$: $u_n = f(n) \mod m$

  $$f(n) = \sum_{j \geq 0} f(\varepsilon_j(n)) \quad \text{and} \quad f(0) = 0.$$ 

- **q-block-additive function** modulo $m$: $u_n = f(n) \mod m$

  $$f(n) = \sum_{j \geq 0} f(\varepsilon_j(n), \varepsilon_{j+1}(n), \ldots, \varepsilon_{j+k-1}(n)) \quad \text{and} \quad f(0, 0, \ldots, 0) = 0.$$
Automatic sequences

- For every $q$-automatic sequence $u_n$ (on an alphabet $\mathcal{A}$) there exists the **logarithmic density** (for every letter $a \in \mathcal{A}$)

  $$\text{logdens}(u_n, a) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{1 \leq n \leq N} \frac{1}{n} \cdot I[u_n = a]$$

  which is also computable.

- If the **densities**

  $$\text{dens}(u_n, a) = \lim_{N \to \infty} \frac{1}{N} \#\{n \leq N : u_n = a\}$$

  exist then they coincide with the logarithmic densities.

- Every **subsequence** $u_{an+b}$ along an **arithmetic progression** of an automatic sequence $u_n$ is automatic, too.

- The **subword complexity** $p_k$ of an automatic sequence is (at most) **linear**.
Subsequences of Automatic Sequences

General idea:
1. Start with an automatic sequence $u_n$ that is uniformly distributed on the output alphabet.
   (Recall: $u_n$ has at most linear subword complexity)
2. Consider a relatively sparse subsequence $u_{n_k}$ that has the same asymptotic frequencies.
   (It is assumed that the average size of the gaps increases sufficiently fast so that one can expect random properties)
3. This subsequence should be pseudo-random (or normal) on the output alphabet.
Thue-Morse sequence along Piatetski-Shapiro sequence $[n^c]$:

Thue-Morse sequence $(t_n)_{n \geq 0}$:

$$011010011001011010011001011010011001011001101 \cdots$$
Thue-Morse sequence along Piatetski-Shapiro sequence $\lfloor n^c \rfloor$

Thue-Morse sequence $(t_n)_{n \geq 0}$:

$$0110100110010110100101100110100110010110011011\ldots$$
Thue-Morse sequence along Piatetski-Shapiro sequence $\lfloor n^c \rfloor$

Thue-Morse sequence $(t_n)_{n \geq 0}$:

\[
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \ldots
\end{array}
\]

Mauduit and Rivat (1995, 2005): $1 < c < 4/3$, $1 < c < 7/5$, Spiegelhofer (2014, 2015+), $1 < c < 1.42$, $1 < c < 1.5 \Rightarrow$

\[
\# \{0 \leq n < N : t_{\lfloor n^c \rfloor} = 0 \} \sim \frac{N}{2}
\]
Subsequences along $\lfloor n^c \rfloor$

**Theorem (Deshouillers, D. and Morgenbesser, 2012)**

Let $u_n$ be a $q$-automatic sequence (on an alphabet $\mathcal{A}$) and $1 < c < 7/5$.

Then for each $a \in \mathcal{A}$ the asymptotic density $\text{dens}(u_{\lfloor n^c \rfloor}, a)$ of $a$ in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of $\alpha$ in $u_n$ exists and we have

$$\text{dens}(u_{\lfloor n^c \rfloor}, a) = \text{dens}(u_n, a).$$

The same property holds for the logarithmic density.
Thue-Morse sequence along squares

Thue-Morse sequence \((t_n)_{n \geq 0}\):

011010011001011010010110011010011001011001101\ldots
Thue-Morse sequence along squares

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
011010011001011010010110011010011001011001101 \ldots
\]
Thue-Morse sequence along squares

Thue-Morse sequence \( (t_n)_{n \geq 0} \):

\[
01 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad \ldots
\]

Mauduit and Rivat (2009):

\[
\# \{ 0 \leq n < N : t_n^2 = 0 \} \sim \frac{N}{2}
\]
Thue-Morse sequence along squares

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
01 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ \ldots
\]

Mauduit and Rivat (2009):

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\]

Solution of a **Conjecture of Gelfond** (1968)
Theorem (Müllner, 2016+)

Let $u_n$ be a $q$-automatic sequence (on an alphabet $A$) generated by a strongly connected automaton such that a zero input at the initial state is mapped to the initial state. Then for each $a \in A$ the asymptotic density

$$\text{dens}(u_{n^2}, a)$$

exists (and can be computed).
Subsequences along squares

Theorem (Müllner, 2016+)

Let $u_n$ be a $q$-automatic sequence (on an alphabet $A$) generated by a strongly connected automaton such that a zero input at the initial state is mapped to the initial state. Then for each $a \in A$ the asymptotic density

$$\text{dens}(u_{n^2}, a)$$

exists (and can be computed).

This also generalizes a result of D.+Morgenbesser (2012) on invertible automatic sequences, where the transitions on the automaton are invertible. The proof is based on a clever representation of automatic sequences and relies very much on a general method by Mauduit and Rivat (2015+) that was applied to the Rudin-Shapiro sequence.
Thue-Morse sequence along primes

Thue-Morse sequence \((t_n)_{n \geq 0}:\)

011010011001011010010110011010011001011001101 \cdots
Thue-Morse sequence along primes

Thue-Morse sequence \((t_n)_{n \geq 0}\):

011010011001011010010110011010011001011001101 \ldots
Thue-Morse sequence along primes

Thue-Morse sequence \((t_n)_{n \geq 0}\):

10 0 1 1 1 0 1 0 0 1 1 1 0 \ldots

Mauduit and Rivat (2010):

\[
\# \{0 \leq p < N : t_p = 0\} \sim \frac{\pi(N)}{2}
\]
Thue-Morse sequence along primes

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
10\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ \ldots
\]

Mauduit and Rivat (2010):

\[
\# \{0 \leq p < N : t_p = 0\} \sim \frac{\pi(N)}{2}
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Solution of a **Conjecture of Gelfond** (1968)
Thue-Morse sequence along primes

Thue-Morse sequence $(t_n)_{n \geq 0}$:

\[
10 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad \ldots
\]

Mauduit and Rivat (2010):

\[
\# \{0 \leq p < N : t_p = 0\} \sim \frac{\pi(N)}{2}
\]

Solution of a **Conjecture of Gelfond** (1968)
Related to the **Sarnak Conjecture**
Theorem (Müllner, 2016+)

Let $u_n$ be a $q$-automatic sequence (on an alphabet $A$) generated by a strongly connected automaton such that a zero input at the initial state is mapped to the initial state. Then for each $a \in A$ the asymptotic density

$$\text{dens}(u_{p_n}, a)$$

exists, where $p_n$ denotes the $n$-th prime number.
Subsequences along primes

**Theorem (Müllner, 2016+)**

Let $u_n$ be a $q$-automatic sequence (on an alphabet $A$) generated by a strongly connected automaton such that a zero input at the initial state is mapped to the initial state. Then for each $a \in A$ the asymptotic density

$$\text{dens}(u_{p_n}, a)$$

exists, where $p_n$ denotes the $n$-th prime number.

This also generalizes a result of D. (2014) on invertible automatic sequences.
Sarnak conjecture for automatic sequences

**Theorem (Müllner, 2016+)**

Let $u_n$ be a complex valued $q$-automatic sequence. Then we have

$$\sum_{n<N} \mu(n) u_n = o(N),$$

where $\mu(n)$ denotes the Möbius function.
Theorem (Müllner, 2016+)

Let $u_n$ be a complex valued $q$-automatic sequence. Then we have

$$\sum_{n<N} \mu(n)u_n = o(N),$$

where $\mu(n)$ denotes the Möbius function.

This generalizes several results by Dartyge and Tenenbaum (Thue-Morse); Mauduit and Rivat (Rudin-Shapiro); Tao (Rudin-Shapiro); D. (invertible); Ferenczi, Kułaga-Przymus, Lemanczyk, and Mauduit (invertible); Deshouillers, D. and Müllner (synchronizing).
Thue-Morse sequence along squares

\[ p_k^{(2)} \] ... subword complexity of \((t_{n^2})_{n \geq 0}\).

Conjecture (Allouche and Shallit, 2003)

\[ p_k^{(2)} = 2^k \]

Equivalently: every block \( B \in \{0, 1\}^k, k \geq 1 \), appears in \((t_{n^2})_{n \geq 0}\).

[Moshe, 2007]: \( p_k^{(2)} = 2^k \)

**Problem.** What can be said about the frequency of a given block?
Thue-Morse sequence along squares

Definition

A sequence $(u_n)_{n \geq 0} \in \{0, 1\}^\mathbb{N}$ is normal if for any $k \in \mathbb{N}$ and any $B = (b_0, \ldots, b_{k-1}) \in \{0, 1\}^k$, we have

$$\lim_{N \to \infty} \frac{1}{N} \# \{ i < N, u_i = b_0, \ldots, u_{i+k-1} = b_{k-1} \} = \frac{1}{2^k}.$$
**Thue-Morse sequence along squares**

**Definition**

A sequence \((u_n)_{n \geq 0} \in \{0, 1\}^\mathbb{N}\) is normal if for any \(k \in \mathbb{N}\) and any \(B = (b_0, \ldots, b_{k-1}) \in \{0, 1\}^k\), we have

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ i < N, \ u_i = b_0, \ldots, u_{i+k-1} = b_{k-1} \} = \frac{1}{2^k}.
\]

**Remark.** There are only few (known) explicit examples of normal sequences.
Theorem (D. Mauduit + Rivat 2013+)

The sequence \((t_{n^2})_{n \geq 0}\) is normal.
Normal subsequences

Theorem (Spiegelhofer 2014+, Spiegelhofer+Müllner 2015+)

Suppose that $1 < c < 3/2$. Then the sequence $(t_{\lfloor nc \rfloor})_{n \geq 0}$ is normal.
Normal subsequences

Theorem (Spiegelhofer 2014+, Spiegelhofer+Müllner 2015+)

Suppose that $1 < c < 3/2$. Then the sequence $(t_{\lfloor nc \rfloor})_{n \geq 0}$ is normal.

Theorem (Müllner 2015+)

Let $f(n)$ be a $q$-block-additive function and $u_n = f(n) \mod m$ an automatic sequence with is uniformly distributed on the alphabet $A = \{0, 1, \ldots, m-1\}$.

Then the sequence $(u_{\lfloor nc \rfloor})_{n \geq 0}$ is normal for all $c$ with $1 < c < 4/3$.

Furthermore if the subsequence $(u_{n^2})_{n \geq 0}$ is uniformly distributed on the alphabet $A = \{0, 1, \ldots, m-1\}$ then $(u_{n^2})_{n \geq 0}$ is normal.
★ Normal subsequences

Conjecture (1)

Suppose that $c > 1$ and $c \notin \mathbb{Z}$. Then for every automatic sequence $u_n$ (on an alphabet $A$) the asymptotic density $\text{dens}(u_{\lfloor n^c \rfloor}, a)$ of $a \in A$ in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of $\alpha$ in $u_n$ exists and we have

$$
\lim_{N \to \infty} \frac{1}{N} \# \{ n < N, \ u_{\lfloor n^c \rfloor} = b_0, \ u_{\lfloor (n+1)^c \rfloor} = b_1, \ldots, \ u_{\lfloor (n+k-1)^c \rfloor} = b_{k-1} \} \\
= \text{dens}(u_n, b_0) \cdot \text{dens}(u_n, b_1) \cdot \ldots \cdot \text{dens}(u_n, b_{k-1})
$$

for every $k \geq 1$ and for all $b_0, \ldots, b_{k-1} \in A$. 
Normal subsequences of Automatic Sequences

Conjecture (2)

Let $P(x)$ be a positive integer valued polynomial and $u_n$ an automatic sequence generated by a strongly connected automaton. Then for every $a \in \mathcal{A}$ the densities $\delta_a = \text{dens}(u_{P(n)}, a)$ exist and we have

$$\lim_{N \to \infty} \frac{1}{N} \# \{ n < N, \ u_{P(n)} = b_0, \ u_{P(n+1)} = b_1, \ldots, u_{P(n+k-1)} = b_{k-1} \} = \delta_{b_0} \cdot \delta_{b_1} \cdots \delta_{b_{k-1}}$$

for every $k \geq 1$ and for all $b_0, \ldots, b_{k-1} \in \mathcal{A}$.
Limits of the method

Let $u_n$ be an automatic sequence and $\phi(n)$ a positive sequences such that $\phi(n)/n$ is non-decreasing.

What can be said about $u_{\lfloor \phi(n) \rfloor}$?

- We cannot expect general results for exponentially growing sequences $\phi(n)$.
- If $\phi(n) = an + b$ with integers $a, b$ then $u_{\phi(n)}$ is again an automatic sequence.
- If $\phi(n) = n \log_2 n$ then $t_{\lfloor \phi(n) \rfloor}$ behaves as the Thue-Morse sequence $t_n$ but the limit

$$\lim_{N \to \infty} \frac{1}{N} \# \{n < N, \ t_{\lfloor n \log_2 n \rfloor} = b_0, \ t_{\lfloor (n+1) \log_2 (n+1) \rfloor} = b_1 \}$$

does not exist. [Deshouilliers+D.+Morgenbesser (2012)]
General subsequences

Conjecture (3)

Suppose that $\phi(x)$ is a positive function such that $\log \phi(x) \sim c \log x$ for some $c > 1$ as well as $\phi'(x)/\phi(x) \sim c/x$ and $c_1/x^2 \leq \phi''(x)/\phi(x) \leq c_2/x^2$ (for some constants $c_1, c_2$ that have the same sign).

Then for every automatic sequence $u_n$ (on an alphabet $A$) that is generated by a strongly connected automaton the asymptotic densities

$$\text{dens}(u_{\lfloor \phi(n) \rfloor}, a)$$

and

$$\text{dens}(u_{\lfloor \phi(p_n) \rfloor}, a)$$

of $a \in A$ exist.

(As above $p_n$ denotes the $n$-th prime number.)
Proof methods

- Comparision of $u_n$ and $u_{\lfloor \phi(n) \rfloor}$ by a clever partial summation
- Fourier analytic sieving
- Clever representation of automatic sequences
Proposition (Deshouilliers+D.+Morgenbesser)

Suppose that $u_n$ is a complex valued automatic sequences and $1 < c < 7/5$. Then we have

$$\left| \sum_{n=0}^{N} u_{\lfloor nc \rfloor} - \frac{1}{c} \sum_{n=0}^{N} n_c^{1-c} u_n \right| \ll N^{1-\delta},$$

where $\delta < (7 - 5c)/9$. 
Clever partial summation

**Proposition (Deshouilliers+D. +Morgenbesser)**

Suppose that $u_n$ is a complex valued automatic sequences and $1 < c < 7/5$. Then we have

$$\left| \sum_{n=0}^{N} u_{\lfloor n^c \rfloor} - \frac{1}{c} \sum_{n=0}^{N} n^{c-1} u_n \right| \ll N^{1-\delta},$$

where $\delta < (7 - 5c)/9$.

This generalizes a method by Mauduit and Rivat (2005) and uses Vaaler’s approximation method as well as the double large sieve.
Fourier estimates

Truncated sum-of-digits function

\[ s_{2,\lambda}(n + k2^\lambda) = s_2(n), \quad 0 \leq n < 2^\lambda, \quad k \geq 0. \]

Alternatively

\[ s_{2,\lambda}(n) = \sum_{i=0}^{\lambda-1} \varepsilon_i(n), \]

where

\[ n = \sum_{i=0}^{\infty} \varepsilon_i(n)2^i \quad \varepsilon_i(n) \in \{0, 1\}, \]

\( s_{2,\lambda} \) is periodic with period \( 2^\lambda \)
Fourier estimates

Discrete Fourier transform

\[ F_\lambda(h, \alpha) = \frac{1}{2^\lambda} \sum_{0 \leq u < 2^\lambda} e(\alpha s_{2,\lambda}(u) - hu2^{-\lambda}) \]

of the function \( n \mapsto e(\alpha s_{q,\lambda}(n)) \); \( e(x) = \exp(2\pi ix) \).
Fourier estimates

Discrete Fourier transform

$$F_\lambda(h, \alpha) = \frac{1}{2^\lambda} \sum_{0 \leq u < 2^\lambda} e(\alpha s_{2^\lambda}(u) - hu2^{-\lambda})$$

of the function $n \mapsto e(\alpha s_q(\lambda(n))$; $e(x) = \exp(2\pi ix)$.

$$F_\lambda(h, \alpha) = \frac{1}{2^\lambda} \prod_{0 \leq k < \lambda} \left(1 + e\left(\alpha - h2^{k-\lambda}\right)\right)$$
Lemma

\( \varphi(x) := 1 + e(x) \quad \Longrightarrow \quad \max_{0 \leq x < 1} |\varphi(\alpha - x)\varphi(\alpha - 2x)| \leq 4 e^{-c\|\alpha\|^2}. \)

for some constant \( c > 0 \). (\( \|\alpha\| = \min\{|\alpha - k| : k \in \mathbb{Z}\} \))
Fourier estimates

Lemma

\[ \varphi(x) := 1 + e(x) \implies \max_{0 \leq x < 1} |\varphi(\alpha - x)\varphi(\alpha - 2x)| \leq 4 e^{-c\|\alpha\|^2}. \]

for some constant \( c > 0 \). (\( \|\alpha\| = \min\{|\alpha - k| : k \in \mathbb{Z}\} \))

Corollary

\[ |F_\lambda(h, \alpha)| \leq 2^{-c\|\alpha\|^2[m/2]} |F_{\lambda-m}(h, \alpha)| \]
Proposition

Suppose that $F_{\lambda}(h, \alpha)$ satisfies the property

$$|F_{\lambda}(h, \alpha)| \leq 2^{-c\|\alpha\|^2 \lfloor m/2 \rfloor} |F_{\lambda-m}(h, \alpha)|$$

(for some $c > 0$. Then it follows that

$$\left| \sum_{n<N} e(\alpha_s_2(n^2)) \right| \ll N^{1-c'\|\alpha\|^2}$$

(for some constant $c' > 0$) and consequently

$$\# \{0 \leq n < N : t_{n^2} = 0 \} \sim \frac{N}{2}$$
Proposition

Suppose that $F_\lambda(h, \alpha)$ satisfies the property

$$|F_\lambda(h, \alpha)| \leq 2^{-c\|\alpha\|^2 \lfloor m/2 \rfloor} |F_{\lambda-m}(h, \alpha)|$$

(for some $c > 0$. Then it follows that

$$\left| \sum_{n<N} e(\alpha s_2(n^2)) \right| \ll N^{1-c'\|\alpha\|^2}$$

(for some constant $c' > 0$) and consequently

$$\# \{0 \leq n < N : t_{n^2} = 0 \} \sim \frac{N}{2}$$

Proof methods: two applications of the Van-der-Corput inequality, a proper Fourier analysis and estimates for quadratic exponential sums.
Fourier estimates

Fourier term with correlations in order to handle blocks of length $> 1$:

$$G^l_{\lambda}(h, d) = \frac{1}{2^\lambda} \sum_{0 \leq u < 2^\lambda} e \left( \frac{1}{2} \sum_{\ell=0}^{k-1} \alpha_{\ell} s_{2,\lambda}(u + \ell d + i_{\ell}) - hu2^{-\lambda} \right),$$

where $\alpha_0, \ldots, \alpha_{k-1} \in \{0, 1\}$ and $l = (i_0, \ldots, i_{k-1}) \in \mathcal{I}_k$:

$$\mathcal{I}_k := \{ l = (i_0, \ldots, i_{k-1}) : i_0 = 0, i_{\ell-1} \leq i_{\ell} \leq i_{\ell-1} + 1, 1 \leq \ell \leq k - 1 \}$$
Fourier estimates

Fourier term with correlations in order to handle blocks of length $> 1$:

$$G^l_\lambda(h, d) = \frac{1}{2^\lambda} \sum_{0 \leq u < 2^\lambda} e\left(\frac{1}{2} \sum_{\ell=0}^{k-1} \alpha_\ell s_{2, \lambda}(u + \ell d + i_\ell) - hu 2^{-\lambda}\right),$$

where $\alpha_0, \ldots, \alpha_{k-1} \in \{0, 1\}$ and $l = (i_0, \ldots, i_{k-1}) \in \mathcal{I}_k$:

$$\mathcal{I}_k := \{l = (i_0, \ldots, i_{k-1}) : i_0 = 0, i_{\ell-1} \leq i_\ell \leq i_{\ell-1} + 1, 1 \leq \ell \leq k - 1\}$$

Uniform upper bounds.

$$\max_{l \in \mathcal{I}_k} \left| G^l_\lambda(h, d) \right| \ll 2^{-\eta m} \max_{J \in \mathcal{I}_k} \left| G^J_\lambda-m(h, \lfloor d/2^m \rfloor) \right|$$

(for some constant $\eta > 0$ and odd $K = \alpha_0 + \cdots + \alpha_{k-1}$).
Fourier estimates

**Proposition**

Suppose that $G^l_\lambda(h, d)$ satisfies the property

$$\max_{l \in I_k} \left| G^l_\lambda(h, d) \right| \ll 2^{-\eta m} \max_{J \in I_k} \left| G^J_{\lambda-m}(h, \lfloor d/2^m \rfloor) \right|$$

(for some $\eta > 0$ and odd $K$). Then it follows that

$$\sum_{n < N} e\left( \frac{1}{2} \sum_{\ell=0}^{k-1} \alpha_\ell s_2((n + \ell)^2) \right) \ll N^{1-\eta'}$$

for some constant $\eta' > 0$ and odd $K$, where $\alpha_0, \ldots, \alpha_{k-1} \in \{0, 1\}$. 
Proposition

Suppose that $G^\lambda_l(h, d)$ satisfies the property

$$\max_{l \in I_k} \left| G^\lambda_l(h, d) \right| \ll 2^{-\eta m} \max_{J \in I_k} \left| G^\lambda_{J-m}(h, \lfloor d/2^m \rfloor) \right|$$

(for some $\eta > 0$ and odd $K$). Then it follows that

$$\sum_{n < N} e \left( \frac{1}{2} \sum_{\ell=0}^{k-1} \alpha_\ell s_2((n + \ell)^2) \right) \ll N^{1-\eta'}$$

for some constant $\eta' > 0$ and odd $K$, where $\alpha_0, \ldots, \alpha_{k-1} \in \{0, 1\}$.

For even $K$ a corresponding property holds and so we get

$$\# \{0 \leq n < N : t_{n^2} = b_0, \ldots, t_{(n+k-1)^2} = b_{k-1} \} \sim \frac{N}{2^k}.$$
Combination of invertible and synchronizing automata:

**Proposition (Müllner)**

Suppose that $A$ is an automaton such that the input 0 maps the initial state of $A$ to itself and let $u = (u_n)_{n \geq 0}$ be the corresponding automatic sequence.

Then there exists a **synchronizing automaton** $A'$ and **permutation matrices** $M_0, \ldots, M_{q-1}$ such that

$$u_n = f(u'_n, S(n)),$$

where $u'_n$ is the automatic sequence related to $A'$,

$$S(n) = M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

and $f$ is a properly chosen function.
Definition

An automaton \( A \) is called **synchronizing** if there exists a **synchronizing word** \( w_0 \) on the input alphabet such that \( w_0 \) applied to all initial states terminates always in the same state of \( A \).
Synchronizing automatic sequences

**Definition**

An automaton $A$ is called **synchronizing** if there exists a **synchronizing word** $w_0$ on the input alphabet such that $w_0$ applied to all initial states terminates always in the same state of $A$.

synchronizing word = 00
Thank you!