Normal Subsequences of Automatic Sequences

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★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

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Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

0

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

01

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

0110

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

01101001

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

0110100110010110

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

$$01101001100101101001011001101001$$

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Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

$$01101001100101101001011001101001100101100 1101 \cdots$$

$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

$$01101001100101101001011001101001100101100110 1\cdots$$

$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n) \qquad \text{or} \qquad t_{2k} = t_k,\ t_{2k+1} = 1 - t_k$$

$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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$$\#\{0 \leqslant n < N : t_n = 0\} \sim \frac{N}{2}$$

★ Thue-Morse sequence

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$$011010011001011010010110011010011001011001101 \cdots$$

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The letters 0 and 1 appear with asymptotic frequency $\frac{1}{2}$.

★ Thue-Morse sequence

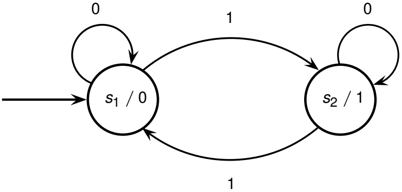
- TM sequence is **not periodic** and **cubeless**.
- TM sequence is **almost periodic**:
  Every appearing consecutive block appears infinitely many times with bounded gaps.
- **Subword complexity is linear**: $p_k \leqslant \frac{10}{3} k$
  p_k ... subword complexity (*number of different consecutive blocks of length k that appear in the TM sequence*).
- **Zero topological entropy** of the corresponding dynamical system:

$$h = \lim_{k\to\infty} \frac{1}{k} \log p_k = 0$$

- **Linear subsequences** $(t_{an+b})_{n\geqslant 0}$ have the same properties.
- The TM sequence and its linear subsequences are **automatic sequences**.

★ Thue-Morse sequence

Automaton that generates the Thue-Morse sequence:
$t_n = \sum_{j \geqslant 0} \varepsilon_j(n) \bmod 2$



★ Rudin-Shapiro sequence

Rudin-Shapiro sequence $(r_n)_{n\geqslant 0}$:

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$$0001001000011101000100101110001000010010000111011111\cdots$$

★ Rudin-Shapiro sequence

Rudin-Shapiro sequence $(r_n)_{n\geqslant 0}$:

$$0001001000011101000100101110001000010010000111011 11\cdots$$

$$r_0 = 0, \qquad r_{2k} = r_k, \qquad r_{2k+1} = \begin{cases} r_k & \text{if } k \text{ is even,} \\ 1 - r_k & \text{if } k \text{ is odd.} \end{cases}$$

★ Rudin-Shapiro sequence

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$$\boxed{r_n = \sum_{i\geqslant 0} \varepsilon_i(n)\varepsilon_{i+1}(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}$$

★ Rudin-Shapiro sequence

Automaton that generates the Rudin-Shapiro sequence:
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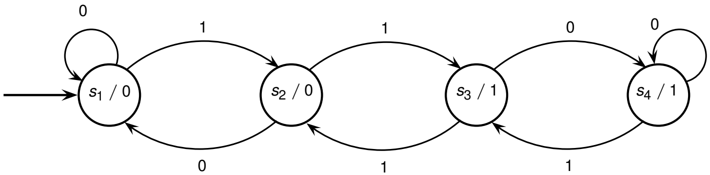
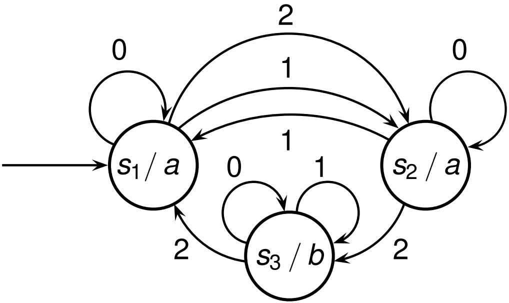


★ Automatic sequences

Definition

A sequence $(u_n)_{n\geqslant 0}$ is called a *q-automatic sequence*, if u_n is the output of an automaton when the input is the q-ary expansion of n.



$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba*...

★ Automatic sequences

- Sum-of-digits-function: $\boxed{u_n = s_q(n) \bmod m}$

- q**-additive function** modulo m: $\boxed{u_n = f(n) \bmod m}$

$$f(n) = \sum_{j \geqslant 0} f(\varepsilon_j(n)) \quad \text{and} \quad f(0) = 0.$$

- q**-block-additive function modulo** m: $\boxed{u_n = f(n) \bmod m}$

$$f(n) = \sum_{j \geqslant 0} f(\varepsilon_j(n), \varepsilon_{j+1}(n), \ldots, \varepsilon_{j+k-1}(n)) \quad \text{and} \quad f(0, 0, \ldots, 0) = 0.$$

★ Automatic sequences

- For every q-automatic sequence u_n (on an alphabet $\mathcal{A}$) there exists the **logarithmic density** (for every letter $a \in \mathcal{A}$)

$$\mathrm{logdens}(u_n, a) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{1\leqslant n\leqslant N} \frac{1}{n} \cdot \mathbf{1}_{[u_n=a]}$$

  which is also computable.

- If the **densities**

$$\mathrm{dens}(u_n, a) = \lim_{N\to\infty} \frac{1}{N} \#\{n \leqslant N : u_n = a\}$$

  exist then they coincide with the logarithmic densities.

- Every **subsequence** u_{an+b} along an **arithmetic progression** of an automatic sequence u_n is automatic, too.
- The **subword complexity** p_k of an automatic sequence is (at most) **linear**.

★ Subsequences of Automatic Sequences

★ **General idea:**

1. Start with an **automatic sequence** u_n that is uniformly distributed on the output alphabet.
(Recall: u_n has at most linear subword complexity)
2. Consider a relatively sparse **subsequence** u_{n_k} that has the same asymptotic frequencies.
(It is assumed that the average size of the gaps increases sufficiently fast so that one can expect **random properties**)
3. This subsequence should be **pseudo-random** (or **normal**) on the output alphabet

★ Thue-Morse sequence along Piatetski-Shapiro sequence $\lfloor n^c \rfloor$

Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

$$0110100110010110100101100110100110010110011010 \cdots$$

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Thue-Morse sequence $(t_n)_{n \geqslant 0}$:

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★ **Thue-Morse sequence along Piatetski-Shapiro sequence $\lfloor n^c \rfloor$**

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

011 10 11 0 11 0 0 0 1 1 1 1 $\cdots$

Mauduit and Rivat (1995, 2005): $1 < c < 4/3$, $1 < c < 7/5$,
Spiegelhofer (2014,2015+), $1 < c < 1.42$, $1 < c < 1.5 \Longrightarrow$

$$\#\{0 \leqslant n < N : t_{\lfloor n^c \rfloor} = 0\} \sim \frac{N}{2}$$

★ Subsequences along $\lfloor n^c \rfloor$

Theorem (Deshouillers, D. and Morgenbesser, 2012)

Let u_n be a q-automatic sequence (on an alphabet $\mathcal{A}$) and

$$1 < c < 7/5.$$

Then for each $a \in \mathcal{A}$ the asymptotic density $\mathrm{dens}(u_{\lfloor n^c \rfloor}, a)$ of a in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of α in u_n exists and we have

$$\boxed{\mathrm{dens}(u_{\lfloor n^c \rfloor}, a) = \mathrm{dens}(u_n, a)}.$$

The same property holds for the logarithmic density.

★ Thue-Morse sequence along squares

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

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01 1 0 1 1 0 $\cdots$

Mauduit and Rivat (2009):

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Solution of a **Conjecture of Gelfond** (1968)

★ Subsequences along squares

Theorem (Müllner, 2016+)

Let u_n be a q-automatic sequence (on an alphabet $\mathcal{A}$) generated by a **strongly connected automaton** *such that a zero input at the initial state is mapped to the initial state.*
Then for each $a \in \mathcal{A}$ the asymptotic density

$$\mathrm{dens}(u_{n^2}, a)$$

exists (and can be computed).

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This also generalizes a result of *D.+Morgenbesser* (2012) on **invertible automatic sequences**, where the transitions on the automaton are invertible. The proof is based on a clever representation of automatic sequences and relies very much on a general method by *Mauduit and Rivat* (2015+) that was applied to the **Rudin-Shapiro sequence**.

★ Thue-Morse sequence along primes

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

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★ Thue-Morse sequence along primes

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

$$10\ 0\ 1\quad 1\ 1\quad 0\ 1\quad 0\quad 0\ 1\quad 1\quad 1\ 0\ \cdots$$

Mauduit and Rivat (2010):

$$\#\{0 \leqslant p < N : t_p = 0\} \sim \frac{\pi(N)}{2}$$

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Solution of a **Conjecture of Gelfond** (1968)
Related to the **Sarnak Conjecture**

★ Subsequences along primes

Theorem (Müllner, 2016+)

Let u_n be a q-automatic sequence (on an alphabet $\mathcal{A}$) generated by a **strongly connected automaton** *such that a zero input at the initial state is mapped to the initial state.*
Then for each $a \in \mathcal{A}$ the asymptotic density

$$\mathrm{dens}(u_{p_n}, a)$$

exists, where p_n denotes the n-th prime number.

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This also generalizes a result of D. (2014) on **invertible automatic sequences.**

★ Sarnak conjecture for automatic sequences

Theorem (Müllner, 2016+)

Let u_n be a complex valued q-automatic sequence.
Then we have

$$\sum_{n<N} \mu(n) u_n = o(N),$$

where $\mu(n)$ denotes the Möbius function.

★ Sarnak conjecture for automatic sequences

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Let u_n be a complex valued q-automatic sequence.
Then we have

$$\sum_{n<N} \mu(n) u_n = o(N),$$

where $\mu(n)$ denotes the Möbius function.

This generalizes several results by *Dartyge and Tenenbaum* (Thue-Morse); *Mauduit and Rivat* (Rudin-Shapiro); *Tao* (Rudin-Shapiro); *D.* (invertible); *Ferenczi, Kułaga-Przymus, Lemanczyk, and Mauduit* (invertible); *Deshoulliers, D. and Müllner* (synchronizing).

★ Thue-Morse sequence along squares

$p_k^{(2)}$ ... subword complexity of $(t_{n^2})_{n\geqslant 0}$.

Conjecture (Allouche and Shallit, 2003)

$$p_k^{(2)} = 2^k$$

Equivalently: every block $B \in \{0,1\}^k$, $k \geqslant 1$, appears in $(t_{n^2})_{n\geqslant 0}$.

[Moshe, 2007]: $p_k^{(2)} = 2^k$

Problem. What can be said about the frequency of a given block?

★ Thue-Morse sequence along squares

Definition

A sequence $(u_n)_{n\geqslant 0} \in \{0,1\}^{\mathbb{N}}$ is normal if for any $k \in \mathbb{N}$ and any $B = (b_0, \ldots, b_{k-1}) \in \{0,1\}^k$, we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{i < N,\ u_i = b_0, \ldots, u_{i+k-1} = b_{k-1}\} = \frac{1}{2^k}.$$

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Remark. There are only few (known) explicit examples of normal sequences.

★ Normal subsequences

Theorem (D.+Mauduit+Rivat 2013+)

The sequence $\boxed{(t_{n^2})_{n\geqslant 0}}$ *is* **normal**.

★ Normal subsequences

Theorem (Spiegelhofer 2014+, Spiegelhofer+Müllner 2015+)

Suppose that $1 < c < 3/2$. *Then the sequence* $\boxed{(t_{\lfloor n^c \rfloor})_{n \geqslant 0}}$ *is* **normal**.

★ Normal subsequences

Theorem (Spiegelhofer 2014+, Spiegelhofer+Müllner 2015+)

Suppose that $1 < c < 3/2$*. Then the sequence* $\boxed{(t_{\lfloor n^c \rfloor})_{n \geqslant 0}}$ *is* **normal**.

Theorem (Müllner 2015+)

Let $f(n)$ *be a* q**-block-additive function** *and* $\boxed{u_n = f(n) \mod m}$ *an automatic sequence with is uniformly distributed on the alphabet* $\mathcal{A} = \{0, 1, \ldots, m-1\}$.
Then the sequence $\boxed{(u_{\lfloor n^c \rfloor})_{n \geqslant 0}}$ *is* **normal** *for all* c *with* $1 < c < 4/3$.
Furthermore if the subsequence $(u_{n^2})_{n \geqslant 0}$ *is uniformly distributed on the alphabet* $\mathcal{A} = \{0, 1, \ldots, m-1\}$ *then* $\boxed{(u_{n^2})_{n \geqslant 0}}$ *is* **normal**.

★ Normal subsequences

Conjecture (1)

Suppose that $c > 1$ and $c \notin \mathbb{Z}$. Then for every automatic sequence u_n (on an alphabet $\mathcal{A}$) the asymptotic density $\mathrm{dens}(u_{\lfloor n^c \rfloor}, a)$ of $a \in \mathcal{A}$ in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of α in u_n exists and we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n < N,\ u_{\lfloor n^c \rfloor} = b_0,\ u_{\lfloor (n+1)^c \rfloor} = b_1, \ldots, u_{\lfloor (n+k-1)^c \rfloor} = b_{k-1}\}$$
$$= \mathrm{dens}(u_n, b_0) \cdot \mathrm{dens}(u_n, b_1) \cdots \mathrm{dens}(u_n, b_{k-1})$$

for every $k \geqslant 1$ and for all $b_0, \ldots, b_{k-1} \in \mathcal{A}$.

★ Normal subsequences

Conjecture (2)

Let $P(x)$ be a positive integer valued polynomial and u_n an automatic sequence generated by a strongly connected automaton.
Then for every $a \in \mathcal{A}$ the densities $\delta_a = \mathrm{dens}(u_{P(n)}, a)$ exist and we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n < N,\ u_{P(n)} = b_0,\ u_{P(n+1)} = b_1, \ldots, u_{P(n+k-1)} = b_{k-1}\}$$
$$= \delta_{b_0} \cdot \delta_{b_1} \cdots \delta_{b_{k-1}}$$

for every $k \geqslant 1$ and for all $b_0, \ldots, b_{k-1} \in \mathcal{A}$.

★ Limits of the method

Let u_n be an automatic sequence and $\phi(n)$ a positive sequences such that $\phi(n)/n$ is non-decreasing.

What can be said about $\boxed{u_{\lfloor \phi(n) \rfloor}}$?

- We cannot expect general results for exponentially growing sequences $\phi(n)$.
- If $\phi(n) = an + b$ with integers a, b then $u_{\phi(n)}$ is again an automatic sequence.
- If $\phi(n) = n \log_2 n$ then $t_{\lfloor \phi(n) \rfloor}$ behaves as the Thue-Morse sequence t_n but the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{n < N,\ t_{\lfloor n \log_2 n \rfloor} = b_0,\ t_{\lfloor (n+1)\log_2(n+1) \rfloor} = b_1\}$$

does **not** exist. [Deshouilliers+D.+Morgenbesser (2012)]

★ General subsequences

Conjecture (3)

Suppose that $\phi(x)$ is a positive function such that $\log \phi(x) \sim c \log x$ for some $c > 1$ as well as $\phi'(x)/\phi(x) \sim c/x$ and $c_1/x^2 \leqslant \phi''(x)/\phi(x) \leqslant c_2/x^2$ (for some constancs c_1, c_2 that have the same sign).
Then for every automatic sequence u_n (on an alphabet $\mathcal{A}$) that is generated by a strongly connected automaton the asymptotic densities

$$\mathrm{dens}(u_{\lfloor \phi(n) \rfloor}, a)$$

and

$$\mathrm{dens}(u_{\lfloor \phi(p_n) \rfloor}, a)$$

of $a \in \mathcal{A}$ exist.
(As above p_n denotes the n-th prime number.)

★ Proof methods

- Comparision of u_n and $u_{\lfloor \phi(n) \rfloor}$ by a *clever* partial summation
- Fourier analytic *sieving*
- Clever representation of automatic sequences

★ Clever partial summation

Proposition (Deshouilliers+D.+Morgenbesser)

Suppose that u_n is a complex valued automatic sequences and $1 < c < 7/5$. Then we have

$$\left| \sum_{n=0}^{N} u_{\lfloor n^c \rfloor} - \frac{1}{c} \sum_{n=0}^{N} n^{\frac{1}{c}-1} u_n \right| \ll N^{1-\delta},$$

where $\delta < (7-5c)/9$.

★ Clever partial summation

Proposition (Deshouilliers+D.+Morgenbesser)

Suppose that u_n is a complex valued automatic sequences and $1 < c < 7/5$. Then we have

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where $\delta < (7-5c)/9$.

This generalizes a method by Mauduit and Rivat (2005) and uses Vaaler's approximation method as well as the double large sieve.

★ Fourier estimates

Truncated sum-of-digits function

$$s_{2,\lambda}(n+k2^\lambda) = s_2(n), \quad 0 \leqslant n < 2^\lambda,\ k \geqslant 0.$$

Alternatively

$$s_{2,\lambda}(n) = \sum_{i=0}^{\lambda-1} \varepsilon_i(n),$$

where

$$n = \sum_{i=0}^{\infty} \varepsilon_i(n) 2^i \qquad \varepsilon_i(n) \in \{0,1\},$$

$s_{2,\lambda}$ is periodic with period 2^λ

★ Fourier estimates

Discrete Fourier transform

$$F_\lambda(h,\alpha) = \frac{1}{2^\lambda} \sum_{0 \leqslant u < 2^\lambda} e(\alpha s_{2,\lambda}(u) - hu2^{-\lambda})$$

of the function $n \mapsto e(\alpha s_{q,\lambda}(n))$; $e(x) = \exp(2\pi i x)$.

★ Fourier estimates

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$$F_\lambda(h,\alpha) = \frac{1}{2^\lambda} \prod_{0\leqslant k<\lambda} \left(1 + e\left(\alpha - h2^{k-\lambda}\right)\right)$$

★ Fourier estimates

Lemma

$\varphi(x) := 1 + e(x) \implies$

$$\max_{0 \leqslant x < 1} |\varphi(\alpha - x)\varphi(\alpha - 2x)| \leqslant 4\, e^{-c\|\alpha\|^2}.$$

for some constant $c > 0$. *(*$\|\alpha\| = \min\{|\alpha - k| : k \in \mathbb{Z}\}$*)*

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Corollary

$$|F_\lambda(h,\alpha)| \leqslant 2^{-c\|\alpha\|^2 \lfloor m/2 \rfloor} |F_{\lambda-m}(h,\alpha)|$$

★ Fourier estimates

Proposition

Suppose that $F_\lambda(h,\alpha)$ *satisfies the property*

$$|F_\lambda(h,\alpha)| \leqslant 2^{-c\|\alpha\|^2 \lfloor m/2 \rfloor} |F_{\lambda-m}(h,\alpha)|$$

(for some $c > 0$*. Then it follows that*

$$\left| \sum_{n<N} e(\alpha s_2(n^2)) \right| \ll N^{1-c'\|\alpha\|^2}$$

(for some constant $c' > 0$*) and consequently*

$$\#\{0 \leqslant n < N : t_{n^2} = 0\} \sim \frac{N}{2}$$

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$$|F_\lambda(h,\alpha)| \leqslant 2^{-c\|\alpha\|^2 \lfloor m/2 \rfloor} |F_{\lambda-m}(h,\alpha)|$$

(for some $c > 0$. Then it follows that

$$\left| \sum_{n<N} e(\alpha s_2(n^2)) \right| \ll N^{1-c'\|\alpha\|^2}$$

(for some constant $c' > 0$) and consequently

$$\# \{0 \leqslant n < N : t_{n^2} = 0\} \sim \frac{N}{2}$$

Proof methods: two applications of the Van-der-Corput inequality, a proper Fourier analysis and estimates for quadratic exponential sums.

★ Fourier estimates

Fourier term with correlations in oder to handle blocks of length > 1:

$$G_\lambda^I(h,d) = \frac{1}{2^\lambda} \sum_{0 \leqslant u < 2^\lambda} \mathrm{e}\left(\frac{1}{2} \sum_{\ell=0}^{k-1} \alpha_\ell s_{2,\lambda}(u + \ell d + i_\ell) - hu2^{-\lambda} \right),$$

where $\alpha_0, \ldots, \alpha_{k-1} \in \{0, 1\}$ and $I = (i_0, \ldots, i_{k-1} \in \mathcal{I}_k$:

$$\mathcal{I}_k := \{I = (i_0, \ldots, i_{k-1}) : i_0 = 0, i_{\ell-1} \leqslant i_\ell \leqslant i_{\ell-1} + 1, 1 \leqslant \ell \leqslant k-1\}$$

★ Fourier estimates

Fourier term with correlations in oder to handle blocks of length > 1:

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Uniform upper bounds.

$$\max_{I \in \mathcal{I}_k} \left| G_\lambda^I(h,d) \right| \ll 2^{-\eta m} \max_{J \in \mathcal{I}_k} \left| G_{\lambda-m}^J(h, \lfloor d/2^m \rfloor) \right|$$

(for some constant $\eta > 0$ and **odd** $K = \alpha_0 + \cdots + \alpha_{k-1}$).

★ Fourier estimates

Proposition

Suppose that $G^I_\lambda(h,d)$ *satisfies the property*

$$\max_{I\in\mathcal{I}_k}\left|G^I_\lambda(h,d)\right| \ll 2^{-\eta m}\max_{J\in\mathcal{I}_k}\left|G^J_{\lambda-m}(h,\lfloor d/2^m\rfloor)\right|$$

(for some $\eta>0$ *and odd* K*). Then it follows that*

$$\sum_{n<N} e\left(\frac{1}{2}\sum_{\ell=0}^{k-1}\alpha_\ell s_2((n+\ell)^2)\right) \ll N^{1-\eta'}$$

for some constant $\eta'>0$ *and odd* K*, where* $\alpha_0,\ldots,\alpha_{k-1}\in\{0,1\}$*.*

★ Fourier estimates

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for some constant $\eta'>0$ and odd K, where $\alpha_0,\ldots,\alpha_{k-1}\in\{0,1\}$.

For even K a corresponding property holds and so we get

$$\#\left\{0\leqslant n<N : t_{n^2}=b_0,\ldots,t_{(n+k-1)^2}=b_{k-1}\right\}\sim\frac{N}{2^k}.$$

★ Representation of automatic sequences

Combination of invertible and synchronizing automata:

Proposition (Müllner)

Suppose that $\mathcal{A}$ is an automaton such that the input 0 *maps the initial state of $\mathcal{A}$ to itself and let $\mathbf{u} = (u_n)_{n \geqslant 0}$ be the corresponding automatic sequence.*
Then there exists a **synchronizing automaton** $\mathcal{A}'$ *and* **permutation matrices** $M_0, \ldots, M_{q-1}$ *such that*

$$\boxed{u_n = f(u_n', S(n))},$$

where u_n' is the automatic sequence related to $\mathcal{A}'$,

$$S(n) = M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

and f is a properly chosen function.

★ Synchronizing automatic sequences

Definition

An automaton $\mathcal{A}$ is called **synchronizing** if there exists a **synchronizing word** w_0 on the input alphabet such that w_0 applied to all initial states terminates always in the same state of $\mathcal{A}$.

★ Synchronizing automatic sequences

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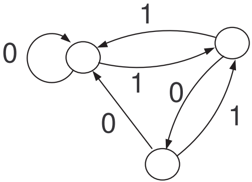

synchronizing word = 00

Thank you!