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Good Banach Limits

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Outline.

1. Completeness of the upper density metric on $\mathbb{N}$

2. Banach limits and Hardy fields

3. Questions on APs in various contexts

4. Quantitave Recurrence, results and counterexamples

SETTINGS.

Let $\mathbb{N}$ stand for the set of natural numbers. Denote by $\mathcal{P} = \mathcal{P}(\mathbb{N})$ the family of subsets of the set $\mathbb{N}$.

Definition (1)

By an *udensity* on $\mathbb{N}$ we mean a map $\phi\colon \mathcal{P} \to [0,1]$ such that:

- $\phi(\mathbb{N}) = 1$; $\phi(\emptyset) = 0$,
- For $A, B \in \mathcal{P}$ we have $A \subset B \Rightarrow \phi(A) \leq \phi(B)$;
- For $A, B \in \mathcal{P}$ we have $\phi(A \cup B) \leq \phi(A) + \phi(B)$.

The set of udensities is denoted by $\mathcal{U} = \mathcal{U}(\mathbb{N})$.

Udensity = upper density = upper measure (on $[0,1]$)

Udensities (upper densities) list

Examples of udensities: (for $A \in \mathcal{P}$)

1. asymptotic udensity: $d^*(A) = \limsup_{n\to\infty} \frac{\left|A\cap[1,n]\right|}{n}$;
   (could be used arbitrary Folner sequence)
2. logarithmic udensity: $d^*_{\log}(A) = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1_A(k)}{k}$;
3. Banach udensity: $d^*_B(A) = \limsup_{n\to\infty} \frac{\left|A\cap[m,m+n]\right|}{n}$;
4. Shnirelman asymptotic (u)density: $d^+(A) = \sup_{n\to\infty} \frac{\left|A\cap[1,n]\right|}{n}$;
5. udensities associated with Hardy fields;
6. means on $\mathbb{N}$: additive (not nec. σ-additive) probabilty measures on $\mathbb{N}$;
7. in particular, ultrafilters: (means valued in the set $\{0,1\}$).

Measurable sets and algebras

A subset $\mathcal{A} \subset \mathcal{P}$ is called a subalgebra if $\emptyset \in \mathcal{A}$ and it is closed under $\cup, \cap$ and taking the complements $(A \in \mathcal{A} \implies A^c := \mathbb{N} \setminus A \in \mathcal{A})$.

Definition

For an udensity $\phi \in \mathcal{U}$, define the collection of ϕ-measurable sets:

$$\mathcal{M}_\phi := \{A \in \mathcal{P} \mid \phi(A) + \phi(A^c) = 1\} \subset \mathcal{P}.$$

Definition

Let an udensity $\phi \in \mathcal{U}$ be given. A subalgebra $\mathcal{A} \subset \mathcal{P}$ is called ϕ-measurable $\phi(A \cup B) = \phi(A) + \phi(B)$ holds for all disjoint subsets $A, B \in \mathcal{A}$. (Equivalently, $(\mathbb{N}, \mathcal{A}, \phi)$ is a measure space (not nec. σ-additive)).

Remark

Remark. Let $\mathcal{A} \subset \mathcal{P}$ be a subalgebra and let $\phi \in \mathcal{U}$ be an udensity. Then:

(1) If $\mathcal{A}$ is measurable, then $\mathcal{A} \subset \mathcal{M} := \mathcal{M}(\phi)$ (every set $A \in \mathcal{A}$ is measurable).

(2) For many reasonable udensities (like d^*, d^*_{log}, d^*_B, but not for all !) the opposite implication also holds: any algebra $\mathcal{A} \subset \mathcal{M}(\phi) \cap \mathcal{P}$ must be measurable.

(3) There are even examples of ϕ for which $\mathcal{M}(\phi) \cap \mathcal{P}$ itself is an algebra:

(a) $([0,1], \lambda^*)$ (upper Lebesgue measure),
(b) means on $\mathbb{N}$,
(c) more complicate.

Udensities as pseudometrics on $\mathcal{P}$

Definition

Every udensity $\phi \in \mathcal{U}$ determines a pseudometric on $\mathcal{P}$ (the same notation is used): $\phi(A,B) = \phi(A \oplus B)$, where $A \oplus B := (A \backslash B) \cup (B \backslash A)$.

We write $A \overset{\phi}{\sim} B$ to denote the relation $\phi(A,B) = 0$.

Note that $(\mathcal{P}, \oplus)$ forms an Abelian group (in fact, a vector space over $\mathbb{Z}_2 = {}^{2\mathbb{Z}}/_{\mathbb{Z}}$), and that $\phi(A, \emptyset) = \phi(A)$. Among the above listed udensities, only d^+ determines (or is) a metric on $\mathcal{P}$.

Theorem (Theorem 1)

$(\mathcal{P}, d^*)$, $(\mathcal{P}, d^*_{\log})$ *are complete pseudometric spaces.* $(\mathcal{P}, d^*_B)$ **is not**.

Completeness of $(\mathcal{P}, d^*)$

Theorem (Theorem 1)

$(\mathcal{P}, d^*)$, $(\mathcal{P}, d^*_{\log})$ are complete pseudometric spaces. $(\mathcal{P}, d^*_B)$ **is not**.

One can define a pseudonorm d^* on the space $L^\infty(\mathbb{N})$ of bounded real sequences.

$$d^*(\mathbf{x}) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{\infty} |x_k|}{n} \quad (\text{where } \ \mathbf{x} = (x_1, x_2, \dots) \in L^\infty(\mathbb{N})).$$

The above Theorem has an equivalent formulation that $(L^\infty(\mathbb{N}), d^*)$ (after the obvious identifications) becomes a Banach space.

Thus $d^*(A) = d^*(1_A)$. Our presentation is with the sets in $\mathbb{N}$ is slightly more convenient.

Useful trivial facts

In what follows, the default udensity ϕ is d^* (if none specified). Thus $A \in \mathcal{M}(d^*)$ is abbreviated to just $\mathcal{M}$ (the collection of measuable sets, i.e. having asymptotic density).

Definition. For $A \in \mathcal{P}$, define $\boxed{d_*(A) = 1 - d^*(A^c)}$, where $A^c := \mathbb{Z} \setminus A$.

Thus $d^*(A) = d_*(A)$ is equivalent to $A \in \mathcal{M} := \mathcal{M}(d^*)$.

Fact 1. $|d^*(A) - d^*(B)| = |d_*(A) - d_*(B)| \leq d^*(A \triangle B), \quad (A, B \in \mathcal{P})$;

Fact 2. $\left(\lim_{n\to\infty} A_n \overset{d^*}{=} A\right) \implies \left(\lim_{n\to\infty} d^*(A_n) = d^*(A) \text{ and } \lim_{n\to\infty} d_*(A_n) = d_*(A)\right)$.

Here $\overset{d^*}{=}$ specifies convergence in metric d^*, and all sets A_n and A lie in $\mathcal{P}$. (In particular, A is a d^*-set if all A_n are).

More useful facts

Fact 3. $\forall k \in \mathbb{N}$ and $A \in \mathcal{P}$, we have $k \in A \;\Rightarrow\; d^+(A) \geq \frac{1}{k}$ and

$$d^+(A) < \frac{1}{k} \;\Rightarrow\; [1,k] \cap A = \emptyset$$

Fact 4. $\forall A \subset \mathbb{N}$, we have $d^*(A) \leq d^+(A)$.

Fact 5. $\forall B \in \mathcal{P}$, $\exists B' \subset B$ such that

1. $B \setminus B'$ is finite (in particular, $d^*(B, B') = d^*(B \triangle B') = 0$);
2. $d^+(B') \leq 2d^*(B) = 2d^*(B')$. (More is true: $\forall \varepsilon > 0$, $\exists B' \subset B$ s. t. $B' \sim B$ and $d^+(B') \leq d^*(B) + \varepsilon$).

We observe that $(\mathcal{P}, d^+)$ is a complete metric space and that the convergence in metric d^+ implies pointwise convergence and convergence in metric d^* (because of Fact 4).

Nice convergence

Definition

Let $(A_k) \in \mathcal{P}^{\mathbb{N}}$ be a sequence of sets and $A \in \mathcal{P}$. Then

1. (A_k) is said to converge to A pointwise (notation $\lim_{k\to\infty} A_k \overset{\text{ptw}}{=} A$) if the pointwise convergence $\lim_{k\to\infty} 1_{A_k} \overset{\text{ptw}}{=} 1_A$ holds.
2. (A_k) is said to converge to A nicely if we have both pointwise convergence and convergence in d^*. (Notation: $\lim_{k\to\infty} A_k \overset{\text{nice}}{=} A$).

Lemma

$(\mathcal{P}, d^+)$ *is a complete metric space. Moreover,*

$$\lim_{k\to\infty} A_k \overset{d^+}{=} A \quad \Longrightarrow \quad \lim_{k\to\infty} A_k \overset{nice}{=} A$$

Proof.

The proof is based on Facts 4 and 5 ($d^* \leq d^+$ and hence specifies weaker topology). ■ □

(The opposite is not true).

A weak σ-additivity result

Next lemma supplies a weak σ-additivity result for the density d^+.

Lemma 2. Let $(A_k) \in \mathcal{P}^{\mathbb{N}}$. Then $d^+\left(\bigcup_{k\geq 1} A_k\right) \leq \sum_{k\geq 1} d^+(A_k)$.

Proof. Straightforward. ∎

Definition A sequence $(A_k) \in \mathcal{P}^{\mathbb{N}}$ is called absolutely summable (rel. udensity ϕ) if $\sum_k \phi(A_k) < \infty$.

Corollary. Let $(A_k) \in \mathcal{P}^{\mathbb{N}}$ be summable rel. d^+. (that is, $\sum_{k\geq 1} d^+(A_k) < \infty$). Then

$$\lim_{N\to\infty} \left(\bigcup_{k=1}^{N} A_k\right) \overset{d^+}{=} \bigcup_{k=1}^{\infty} A_k,$$

and hence the convergence holds also nicely.

In other words, absolute summability in d^+ implies nice summability.

σ-additivity continued

Lemma 3. Let $(A_k) \in \mathcal{P}^{\mathbb{N}}$ be such that $\sum_{k\geq 1} d^+(A_k) < \infty$.
Then two series $\bigcup_{k\geq 1} A_k$ and $\bigoplus_{k\geq 1} A_k$ converges to some $A \in \mathcal{P}(\mathbb{N})$ in the metric d^+ (and hence in the metric d^* and also pointwise).

(In other words, the sequence of partial "sums"

$$B_n = \bigoplus_{k\geq 1}^{n} A_k = A_1 \oplus A_2 \oplus \cdots \oplus A_n$$

converges to some $B \in \mathcal{P}(\mathbb{N})$ in d^+ and hence also nicely).

Proof.

It is enough to show that (B_n) is Cauchy in d^+, and we validate this:

$$d^+(B_n, B_{n+d}) \leq \sum_{k=n}^{n+d-1} d^+(B_k, B_{k+1}) \leq \sum_{k=n}^{n+d-1} d^+(A_{k+1})$$

which becomes small as $n \to \infty$. ■ □

(More general statements are possible)

Completion of the proof

Theorem 1. Pseudometric space $(\mathcal{P}, d^*)$ is complete. That is:

Let $(A_k)_{k\geq 1}$ be a Cauchy sequence in $\mathcal{P}(N)$ relative to the metric d^*.

Then $\exists\, A \in \mathcal{P}(\mathbb{N})$ such that $\lim_{k\to\infty} \mathbf{dist}^*(A_k, A) = 0$.

Proof: Set $B_k = A_k \oplus A_{k+1}$, $s_k = \mathbf{dist}^*(A_k, A_{k+1}) = \mathbf{dist}^*(B_k)$ $(k \geq 1)$.
Since (A_k) is Cauchy, by passing to a subsequence, one can make $\sum_k s_k < \infty$.
Then $A_n = A_1 \oplus \left(\bigoplus_{k=1}^{n-1} B_k\right)$, $\forall\, n \geq 2$. By Fact 5, $\forall\, k \geq 1$ $\exists\, B_k' \sim B_k$ such that $d^+(B_k') \leq 2d^+(B_k') = 2s_k$. Define:

$$A_n' = A_1 \oplus \left(\bigoplus_{k=1}^{n-1} B_k'\right), \qquad \forall\, n \geq 2.$$

Since $\sum_k d^*(B_k) \leq 2 \sum_k s_k < \infty$, by Lemma 3 $\exists A \in \mathcal{P}(\mathbb{N})$ such that $\lim_{n\to\infty} \mathbf{dist}^*(A_n', A) = 0$.

But then $\lim_{n\to\infty} \mathbf{dist}^*(A_n, A) = 0$ because $A_n' \sim A_n$ $(\forall n)$. ■

Even more

Lemma (straightforward)

Let $(A_k)_{k\geq 1}$ *be a monotone (e.g. non-decreasing) sequence of sets* $A_k \in \mathcal{M}(d^*)$ *(having asymptotic density). Then* $\exists A \in \mathcal{M}(d^*)$ *such that* $\lim_{k\to\infty} A_k \overset{d^*}{=} A$, *i. e.* $\lim_{k\to\infty} \mathbf{dist}^*(A_k, A) = 0$.

Proof.

We have only to validate that $(A_k)_{k\geq 1}$ is Cauchy in d^*. Follows from Theorem 1. ■ □

Warning. It is not true in general that $\bigcup_k A_k \sim A$ in the above theorem.

Question. Is it important is that is the requirement that $A_k \in \mathcal{M}(d^*)$?
Answer. It is. Not true without this assumption in general.

Example*. Construct a non-decreasing sequence $(A_k)_{k\geq 1}$ in $\mathcal{P}$ such that $\mathbf{dist}^*(A_k, A_{k+1}) = 1$, $\forall k$. (In fact, there is an ordered by inclusion continuum system of sets (A_t), $t \in [0,1]$, with $\mathbf{dist}^*(A_t, A_s) = 1$ for $t \neq s$).

Generic subalgebras

Definition

A subalgebra $\mathcal{A} \subset \mathcal{P}$ is called:

- (d^*)measurable if $\mathcal{A} \subset \mathcal{M}(d^*)$ i.e., if its every set is measurable;
- shift-invariant if $A \in \mathcal{A} \iff A+1 \in \mathcal{A}$;
- generic if $\mathcal{A}$ is both measurable and shift-invariant.

A set $A \in \mathcal{P}$ is called *generic* if it belongs to an ergodic algebra $(\subset \mathcal{P})$.
Notation: $A \in \mathcal{P}_{gen}$.

Theorem

$A \in \mathcal{P}_{gen}$ if and only if its every block (in 1_A) appears with some asymptotic density (i.e., if 1_A is generic to some measure (called Minsky measure for A)).

Besicovich algebra

Theorem

If $\mathcal{A} \subset \mathcal{P}$ is an algebra, so is its closure $\overline{\mathcal{A}}$. If $\mathcal{A} \subset \mathcal{P}$ is either shift-invariant or generic, the same properties has its closure $\overline{\mathcal{A}}$. In particular, $\overline{\mathcal{A}}$ is generic if $\mathcal{A}$ is.

Basic Example. Denote by $\mathcal{P}_p$ the algebra of periodic subsets of $\mathbb{N}$. It is obviously generic, so is its closure $\overline{\mathcal{P}}_p$, called Besicovich algebra. It contains the sets of square free integers and much more.

Some examples of sets in $\overline{\mathcal{P}}_p$ are provided by the following theorem.

$\mathcal{B}$–free sets of integers generalization

Theorem

Let $z_k = (m_k, n_k)$, $k \geq 1$, *be a sequence of points in* $\mathbb{N}^2$ *(pairs of natural numbers). Then the set* $A = A\big((z_k)\big) := \bigcup_k AP(m_k, n_k)$ *where*

$$AP(m_k, n_k) := \{m_k + n_k d \mid d \geq 0\} = m_k + n_k \mathbb{N}_0; \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

is generic provided the following conditions are met:

1. *The set* $M = \{m_k \mid k \geq 1\}$ *of the first terms of these sequences has density* 0;
2. $\sum_k \frac{1}{n_k} < \infty$.

The above theorem generalizes some result on $\mathcal{B}$–free sets of integers initiates in the first part of the previous century (by Besicovitch and Chowla) and continued by many authors (incl. Davenport and Erdos - reg. condition 2).

Good survey of earlier results - by Ahlswede and Khachatrian.

Recent work - by our organizer M. Lemanczyk, J. Kulaga-Przymus, A. Bartnicka, J. Kasjan $\mathcal{B}$-free sets and dynamics, with many references.

Proof

Proof.

Observe that $A = A' \cup M$ where $A' := \bigcup_k AP(m_k + n_k, n_k)$.

We have $d(M) = d^*(M) = 0$ while A' has density because

$d^+(AP(m_k + n_k, n_k) = \frac{1}{n_k}$, and hence $\bigcup_{k=1}^{N} AP(m_k + n_k, n_k)$

is Cauchy in d^+ and hence is nicely convergent to a set in $\overline{\mathcal{P}}_p$.

(See Lemmas 1, 3 and Basic Example following Theorem 2). ■ □

The above approach allows to validate the densities of various sequences (computed empirically) by approximation these sequences with periodic or other sequences with known densities. Similar arguments can be used in some other situations.

More

Theorem

For any subset of integers $M \subset \mathbb{N}$ the set $\bigcup_{m \in M} m\mathbb{N}^2$ has asymptotic density in $\mathbb{Z}^2$ (and is in fact generic).

One can also state a version with sliding dilations of $\mathbb{N}^k$, $\mathbb{Z}^k$ $(k \geq 1)$.

More Examples. The sets $A \in \mathcal{M}$ whose densities $(\delta(A) = d^*(A) = d_*(A))$ can be computed by approximation with periodic sets or otherwise by approximation:

1. The set $\mathcal{R}$ of square free integers, lies in $\overline{\mathcal{P}}_p \subset \mathcal{M}$.
2. For any polynomial $f(x) \in \mathbb{Z}[x]$, the set
$$\Gamma(f) := \{n \in \mathbb{N} \mid f(n) \in \mathcal{R}\} \subset \overline{\mathcal{P}}_p \subset \mathcal{M}$$
3. For any Bergelson-Leibman generalized polynomial $f(x)$ the set
$$\{n \in \mathbb{N} \mid gcd(n, [f(n)]) \in \mathcal{R}\}.$$
This density is $\frac{1}{\zeta(2)}$ if $f(n) = n$, and it is $\frac{1}{\zeta(4)}$ if $f(x)$ is a polynomial with at least one coefficient (not the free term) irrational.
4. The same as **3.** with $f(x)$ lying in a Hardy field, growing not faster than polynomials and which is u.d. (mod 1).
(There are precise conditions on its growth).

Question

Question. For various functions $f(x)$ confirm that the set

$$\Gamma(f) := \{n \in \mathbb{N} \mid f(n) \in \mathcal{R}\} \in \mathcal{M} \qquad (*)$$

(has asymptotic density). The case $f(x) = \alpha x + \beta$ is easy.

Harder for $f(x) = x^{1.1}$ or $f(x) = \alpha x^2$, with an irrational α.

Lemma on unions

Lemma 5. Let (A_k) be a sequence of measurable (= having asymptotic density) disjoint sets: $A_k \in \mathcal{M} = \mathcal{M}(d^*) \subset \mathcal{P}$. Then there are partitions $A_k = A'_k \cup A''_k$, $A'_k \cap A''_k = \emptyset$ such that, under the notations

$$U := \bigcup_{k=1}^{\infty} A_k, \qquad U_n := \bigcup_{k=1}^{n} A_k,$$

$$U' := \bigcup_{k=1}^{\infty} A'_k, \qquad U'_n := \bigcup_{k=1}^{n} A'_k,$$

$$U'' := \bigcup_{k=1}^{\infty} A''_k, \qquad U''_n := \bigcup_{k=1}^{n} A''_k,$$

the following conditions are met:

1. The sets A''_k are finite and $d^+(A'_k) \leq 2d^*(A_k)$,
2. The sequence (U'_n) converge in d^+, and hence $\lim_{n\to\infty} U'_n \overset{nice}{=} U'$;
3. The set U' is measurable ($U' \in \mathcal{M}$);
4. If moreover the relation $\lim_{n\to\infty} (U_n) \overset{d^*}{=} U$ holds then the set U'' has density 0, and hence $U \sim U'$.

Proof

Proof.

(1.) has been already proved (Step 5).

(2.) follows because $\sum d^+(A_k') = 2(\sum d^*(A_k)) \leq 2$ (see Lemma 1).

(2.) implies (3.). Finally, nice convergence in (2.) implies $\lim_n U_n' \stackrel{d^*}{=} U'$; and, since $U_n \sim U_n'$, the relation $U \sim U'$ follows. The last relation implies that $U'' = U \setminus U'$ has density 0. ■ □

next

Settings for Quantitave Recurrence results

SETTINGS.

1. By an m.p.s. (measure preserving system) $(X, \mathcal{B}, \mu, T)$ we mean a probability measure space $(X, \mathcal{B}, \mu)$ together with a measure preserving map $T\colon X \to X$.

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2. By an m.m.p.s. (metric m.p.s.) $(X, \mathcal{B}, \mu, T, d)$ we mean an m.p.s. $(X, \mathcal{B}, \mu)$ together with a *compatible* metric d on X (such that open subsets in the metric space (X, d) are measurable).

   Note that T is not assumed to be continuos on (X, d)).

Poincare Pointwise Recurrence Theorem - next page

Pointwise Recurrence Theorem

Theorem (PPRT: Poincare Pointwise Recurrence Theorem)

Let $(X, \mathcal{B}, \mu, T, d)$ *be an m.m.p.s. and assume that the metric space* (X, d) *is separable. Then* μ*-a.a. points* $x \in X$ *are recurrent:*

$$\liminf_{n\to\infty} d(x, T^n x) = 0. \qquad \text{(PPRT)}$$

What about the speed of convergence to 0?

Theorem (MB, 1993)

Let (X, d) *be a metric space,* $\alpha > 0$ *and* $\beta = 1/\alpha$*. Then*

1. *If* $H_\alpha(X) = 0$ *then for any m.m.p.s.* $(X, \mathcal{B}, \mu, T, d)$
$$\liminf_{n\to\infty} n^\beta\, d(x, T^n x) = 0, \quad \textit{for a.a. } x \in X.$$
2. *If* $H_\alpha(X) < \infty$ *then for any m.m.p.s.* $(X, \mathcal{B}, \mu, T, d)$
$$\liminf_{n\to\infty} n^\beta\, d(x, T^n x) < \infty, \quad \textit{for a.a. } x \in X.$$

Proved independently by N. G. Moshchevitin in 1998.

Various extensions/generalizations/applications:

Barreira and Saussol (2001,2003), I. Shkredov (6 papers),
N. Moshchevitin (2 papers), D. Kleinbock, J. Chaika, M. Einsiedler,
Kim Dong Han (6 papers, incl. general group actions), and others.

Special case: $X = [0,1]$. (Much easier than theorems in the previous slide). An exercise in Katok-Hassenblatt's book.

Denote by λ the Lebesgue measure on $[0,1]$.

Theorem (MB, 1993)

Let $([0,1], \lambda, T)$ *be a m.p.s. Then* $\boxed{\liminf_{n\to\infty} n \cdot |x - T^n x| \leq 1}$, *for* λ-*a.a.* $x \in [0,1]$. *(An exercise in Katok-Hassenblatt's book).*

QUESTION. What is the best constant? (1993-2015).

Quantitative Recurrence on $[0, 1]$, *new*

Theorem (Vincent Delecroix and MB, 2015)

If T *is a* λ*-preserving transformation of* $[0, 1]$ *then*

$$\liminf_{n\to\infty} n \cdot |x - T^n x| \leq \frac{1}{\sqrt{5}} \quad \textit{(for a.a. } x \in X).$$

In other words, the optimal constant is $\frac{1}{\sqrt{5}}$.

The above theorem extends the classical Hurwitz's result pertaining to rotations of the unit circle to all Lebesgue measure preserving transformations.

Uniform version

Theorem (2016)

Let (X, d) *be a metric space,* $\alpha > 0$ *and* $\beta = 1/\alpha$.

If the upper box dimension of (X, d) *is* $\alpha \in (0, \infty)$ *(of finite* α*-volume*) then for a.a.* $x \in X$

$$\lim_{n\to\infty} \frac{n^\beta}{\log^2 n} \min_{1\le k\le n} d(x, T^k x) = 0.$$

Remarks:

(1) This is a uniform version: Here we take $\lim$ *versus* $\liminf$.

(2) Factor $n^\beta N$ *would not work in general.*

(Already for $x \to 2x \pmod 1$ *on* $[0, 1)$*).*

(3) I believe that with the Hausdorff dimension the above theorem does not allow quantitative version.*

next