# Ergodic theory for Hénon-like maps

Michael Benedicks

KTH Royal Institute of Technology

Ergodic theory and its Connections
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Let us start with a simple dynamical system.

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mathrm{mod}(1)$$

It is a map of the two-torus $\mathbb{T}^2$ onto itself. Since the map is areapreserving. Lebesgue measure $\lambda$ is invariant and Birkhoff's ergodic theorem gives that for almost all initial points $z$ and for any continuous function $\varphi$.

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j z) \to \int_{\mathbb{T}^2} \varphi(x) d\lambda$$

This is an instance of Boltzmann's principle that "time averages" should be equal to "ensamble averages".

The dynamical system given by $A$ as a torus map is one of the simplest examples of what is called an Anosov System.
Let $f : M \mapsto M$ be a diffeomorphism of a compact Riemannian surface.

(a) $f$ is an Anosov diffeomorphism if the tangent space for every $x \in M$, the tangent space $TM_x$ can be split as

$$TM_x = E^u(x) \oplus E^s(x).$$

$E^u(x)$ and $E^s(x)$ are $Df(x)$ invariant subspaces.

$Df|_{E^u}$ is uniformly expanding and $Df|_{E^s}$ is uniformly contracting, i.e there exist constants $\lambda > 1$ and $0 < \mu < 1$ so that

$$|Df|_{E_u} v| \geq \lambda |v|, \quad \forall v \in E_u$$
$$|Df|_{E_s} v| \leq \mu |v|, \quad \forall v \in E_s.$$

(b) A compact invariant set $\Lambda \subset M$ is called and attractor if there is a neighborhood $U \supset \Lambda$ so that

$$\text{dist}(f^n x, \Lambda) \to 0 \quad \text{as } n \to \infty$$

$B = \bigcup_{j \geq 0} f^{-j}(U)$ is called the basin of $\Lambda$.
$\Lambda$ is called an Axiom A attractor if $TM_x$ is split as above for all $x \in \Lambda$.

### Theorem

*(Sinai 1972, Ruelle 1976, 1978, Bowen 1975)*
*Suppose $f$ is $C^2$-diffeomorphism with an Axiom A attractor $\Lambda$. Then there is a unique $f$-invariant Borel probability measure $\mu$*

(i) *$\mu$ has absolutely continuous conditional measures on unstable manifolds*

(ii) *The metric entropy $h_\mu(f)$ can be written as*

$$h_\mu(f) = \int \log \det(Df|_{E^u})\, d\mu.$$

(iii) *There is a set $V \subset U$ of full Lebesgue measure so that for all test functions $\varphi$*

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j x) \to \int \varphi(x)\, d\mu(x)$$

### Theorem

*(Cont.)*

(iv) *$\mu$ is the zero noise limit of $f$ when it is perturbed with a small perturbation.*
*A more precise meaning of the zero noise limit that appears in (iv) can be made as follows: We consider a family of Markov chains $P^\varepsilon(\cdot|\cdot)$, $\varepsilon > 0$, whose densities have some regularity properties. The density of $P^\varepsilon(\cdot|x)$ is concentrated in a ball $B(fx, \varepsilon)$ and we let $\varepsilon \to 0$, and then the invariant measure of the Markov chain converges to $\mu$.*

The origin of SRB-measures comes from statistical mechanics. In 1968, Sinai constructed for Anosov Diffeomorphisms certain partitions called Markov Partitions. These partitions enabled him to identify points in the phase space with configuration in one-dimensional lattice systems. One can also view these lattice configuration as symbol sequences in information theory.

Consider the torus map

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod(1)$$

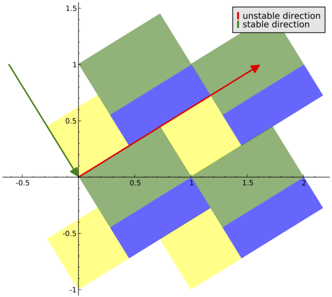



Figure : Markov partition of torus map

We divide the torus into three sets: $A_0$, green region, $A_1$ blue region, and $A_2$ yellow region. The corresponding symbols are 0, 1 and 2.
To each point $(x, y)$ on the torus is associated a sequence

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto (s_0, s_1, s_2, \dots),$$

where the symbol $s_j$ corresponds to that $f^j(x) \in A_{s_j}$.
This makes it possible to use Shannon's definitions of entropy in information theory.

In 1972 Sinai developed for Anosov systems a Gibbs theory analogous to that in Statistical Mechanics. SRB measures are special cases of Gibbs measures. They are defined by the potential $-\log|\det(Df|_{E^u})|$ or equivalently by the fact that they have smooth conditional measures on unstable manifolds. At about the same time Bowen extended the construction of Markov partitions to Axiom A attractors.

A classical attractor with these properties is the so called Solenoid constructed by Smale and Williams.

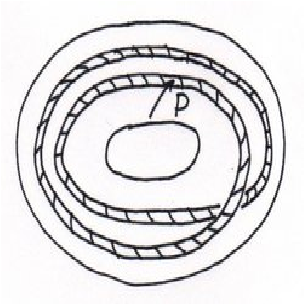



Figure : The Solenoid 1

Figure : The Solenoid 2

We recall the notion of Lyapunov exponents. If $x \in M$ and $v$ is a vector in the tangent space $TM_x$, let

$$\lambda(x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log ||Df_x^n v||$$

if these limits exist and coincide.

A theorem due to Osceledets states that if $\mu$ is an $f$-invariant probablility measure on $M$, then there exist measurable functions $\lambda_i$ such that at $\mu$-a.e. $x$, the tangent space $TM_X = \oplus E_i(x)$ where $\lambda(x, v) = \lambda_i(x)$ for $v \in E_i(X)$. The $\lambda_i$ are called the Lyapunov exponents of $(f, \mu)$.

Suppose now that $(f, \mu)$ has a positive (respectively negative) Lyapunov exponent a.e. Then unstable (resp. stable) manifolds are well defined a.e. More precisely, for $x \in M$, let the unstable manifold be defined by

$$W^u(x) = \{y \in M : \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) < 0\}$$

A well-known theorem by Y. Pesin states that $\mu$-a.e. $x$, $W^u(x)$ is an immersed submanifold tangent at $x$ to $\oplus_{\lambda_i>0}E_i(x)$. The stable manifold at $x$, denoted $W^s(x)$, is defined analogously with $d(f^{-n}x, f^{-n}y)$ in the definition of $W^u$ replaced by $d(f^n x, f^n y)$.

A measure $\mu$ has smooth conditional measures on unstable manifolds if $\mu$ can be disintegrated as

$$\mu = \int \mu_\alpha A(d\alpha)$$

The support of $\mu_\alpha$ is on a local $W^h(z_\alpha)$ leave and it is absolutely continuous.

### Theorem

*(Ruelle 1976, Pesin 1977, Ledrappier and Young 1985). Let $f$ be an arbitrary diffeomorphism and $\mu$ and $\mu$ a $f$-invariant probability measure with a $\mu$ an $f$-invariant Borel probablility measrue with a positive lyapunov exponent a.e. Then $\mu$ has absolutely continuous conditional measures of $W^u$ if and only if*

$$h_\mu(f) = \int \Big( \sum_{\lambda_i > 0} \lambda_i \dim E_i \Big)\, d\mu$$

The inequality $\leq$ is due to Ruelle and true in more generality.

Construction of SRB-measures on Axiom A attractors.
The following is a version of Sinai's original construction.
Let $f : M \mapsto M$ be a $C^2$ diffeomorphism of a compact Riemannian manifold and let $\Lambda \subset M$ be an attractor. We pick an arbitrary piece of local unstable manifold $\gamma \subset \Lambda$, and let $m_\gamma$, denote Lebesgue measure on $\gamma$. Now use $f^i$ to transport $m_\gamma$ forward and denote the image measures by $f^i(m_\gamma)$, $i = 0, 1, 2, \ldots$. We claim that any limit of

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} f_*^i(m_\gamma) \right\}_{n=1,2,\ldots}$$

is an invariant measure.
Invariance is obvious because an average is taken. Since $f^i|_\gamma$ expands distances uniformly, we have bounded distorsion, and $f_*^i(m_\gamma)$ has a density with respect to Lebesgue measure on $f^i(\gamma)$. These densities have uniform upper and lower bounds independent on $i$ and are also aligned with $W^u$.

A lot of the development is based on the study of concrete examples. In some sense the simplest example is the quadratic or logistic map $q_a(x) = 1 - ax^2$, where $x \in [-1, 1]$, $a \in (0, 2]$

**Theorem**

*(Jakobson, Graczyk-Swiatek, Lyubich) The following holds for the logistic family $q_a(x) = 1 - ax^2$, $x \in [-1, 1]$*

(i) *There is an open and dense set $\mathcal{A}$ in parameter space such that for all $a \in \mathcal{A}$, $q_a$ has a periodic sink to which the orbit of Lebesgue-a.e. point converges.*

(ii) *There is a positive Lebesgue measure set of parameters $\mathcal{B}$ such that for $a \in \mathcal{B}$, $q_a$ has an invariant measure absolutely continuous with respect to Lebesgue measure.*

(iii) *The union of $\mathcal{A}$ and $\mathcal{B}$ has full measure in the parameter space.*

Part (ii) of this theorem can be viewed as a one-dimensional version of SRB-measures and this result is due to M. Jakobson. Another proof was given by Carleson&B, which became important for subsequent work on Hénon maps.

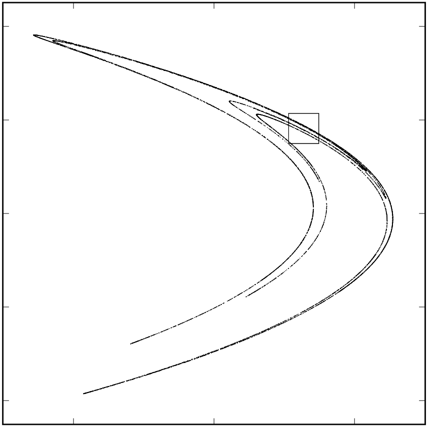
In 1977, M. Hénon carried out numerical studies and also verified rigourously by computations that certain maps had chaotic attractors. The maps he considered were the quadratic maps of the plane with constant determinant given by

$$f_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - ax^2 \\ by \end{pmatrix}$$

Hénon did his computer experiment for $a = 1.4$ and $b = 0.3$. He took an initial point essentially arbitrary and and observed what seemed like a strange attractor, i.e. at least not an attractive sink. What his plots gave can be viewed as empirical observations of the Birkhoff sums.

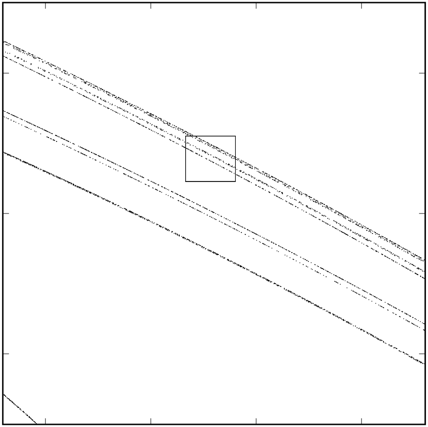
```
P=[1.4,0.3,]
Wx=[0.631354,0.189406,]
nd=10000,it=[0,10000,1],dt=0.01,pol_sec=0,sec_idx=2,sec_c=0,int_op=1

-1.5 -x- 1.5  -0.45 -y- 0.45
```



P=[1.4,0.3,]
Wx=[0.631354,0.189406,]
nd=4141,it=[0,100000,1],dt=0.01,pol_sec=0,sec_idx=2,sec_c=0,int_op=1

0.53 -x- 0.73  0.15 -y- 0.21



## Theorem

*(Carleson& B) Consider the family of Hénon maps*

$$f_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - ax^2 \\ by \end{pmatrix}.$$

*For $b_0 > 0$ and sufficiently small there is for all $b < b_0$ a subset $E_b$ of a-parameters of positive linear Lebesgue measure so that for all $a \in E_b$*

(i) *There is a point $z_0 = z_0(a, b)$ so that the orbit $f = f_{a,b}$ $\{f^n(z_0)\}_{n=0}^{\infty}$ is dense on $\Lambda = \overline{W^u(P_1)}$, where $P_1$ is the fixed point in the first quadrant.*

(ii) *There is a $\lambda = \lambda(a, b) > 0$ so that*

$$|Df^n(z_0)v| \geq e^{\lambda n}, \qquad \text{for} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{say}$$

The difference between the situation for the Axiom A attractors is that no longer we require the transversality of $W^u(x)$ and $W^s(x)$. On the attractor there are points there the stable and unstable manifold are tangent. But the orbit of these points must be controlled. The strong contraction of the unstable manifold at those tangency points the critical points must be compensated by future expansion and the proof goes with a complicated induction. The critical points play in some sense the same role as the critical point 0 for the quadratic family, but the critical points form a a countable set on the unstable manifold and the closure is a Cantor set.

Let us denote the good parameter set in the previous theorem by

$$E = \bigcup_{b<b_0} E_b \times \{b\}$$

For the same parameters L.S. Young and I proved the following theorem

**Theorem**

*For $(a, b) \in E$, $f_{a,b}$ has a unique SRB measure*

As a consequence there is a set $X$ of positive Lebesgue measure in the phase space so that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^z} \to \mu \qquad z \in X.$$

As a consequence of the existence of the SRB-measures we have the following:
The metric entropy of the SRB-measure for Hénon maps with good parameters can be computed as follows:

$$h_\mu(f) = \lambda_1$$

A famous formula due to L.S. Young give

$$\mathrm{HD}(\mu) = h_\mu(f)\left(\frac{1}{\lambda_1} + \frac{1}{|\lambda_2|}\right)$$

Since $e^{\lambda_1} e^{\lambda_2} = b$, we get in this case that

$$\mathrm{HD}(\mu) = 1 + \frac{\lambda_1}{|\lambda_2|} = 1 + \mathcal{O}\left(\frac{1}{\log(1/b)}\right).$$

The following result improves the result by L.S. Young and me on SRB measures for the Hénon maps.

### Theorem

*(Viana& B) For $(a, b) \in E$, $f_{a,b}$ has the Basin Property:*
*For a.e. $z = (x, y)$ in the basin $B = \bigcup_{j \geq 0} f^{-j}(U)$*

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j z} \to \mu \qquad z \in X$$

*holds and $B$ is foliated a.e. with stable leaves $W^s(x)$,so that all leaves hit the attractor $\Lambda$, i.e the basin can be written as*

$$B = \bigcup_{z \in \Lambda} W^u(z)$$

The good parameters $E$ also have the property that small random perturbations have the SRB measure as the zero noise limit. A special case of results of M. Viana and me is the following

**Theorem**

*(Viana&B) Define a Markov chain by*

$$X_{n+1} = f_{a,b}(X_n) + \xi_n,$$

*where $\xi_n$ are iid uniformly distributed random variables on $B(0, \varepsilon)$. Then there is a unique measure $\nu_\varepsilon$ which is stationary for the Markov Chain*

$$\nu_\varepsilon(A) = \int p(A, z) d\nu_\varepsilon(z) \quad \textit{for all } A,$$

*and $\nu_\varepsilon \to \mu$ as $\varepsilon \to 0$.*

Other versions of theory of non-uniformly hyperbolic attractors

Theory of Hénon-like maps (Mora-Viana)

This version of the theory gives information in connection with homoclinic bifurcations. (Palis & Takens)

Theory of attractors with one expanding direction (Wang-Young)

One aspect of the theory of SRB measures is that it can be extended to apply to the study of $(f, \mu)$ as a Stationary Stochastic Process and study decay of correlation etc.

# Tower constructions for unimodal maps

$f : I \mapsto I$, $I = (-1, 1)$, eg. $f(x) = f_a(x) = 1 - ax^2$, $0 < a \leq 2$.
Write a partition

$$(-\delta, \delta) = \bigcup_{|r| \geq r_0} I_r, \qquad I_r = (e^{-r-1}, e^{-r}), \; I_{-r} = -I_r$$

For distorsion purposes we also write

$$I_r = \bigcup_{\ell=0}^{r^2-1} I_{r,\ell}, \qquad |I_{r,\ell}| = \frac{1}{r^2}|I_r|$$

The base of the tower will be the end interval $W = I_{r,r^2-1}$, which we decompose into a partition

$$W = \bigcup_{i \in \Gamma} J_i$$

We want the following Markov properties

$$\begin{cases} f^{n_i}(J_i) = W \\ F|_{J_i}(x) = f^{n_i}(x) \end{cases}$$

We require

$$\begin{cases} \left| \frac{(f^{n_i})'(x)}{f^{n_i})'(y)} \right| \leq \exp\{C|f^{n_i}(x) - f^{n_i}(x)|\}, \qquad C > 0 \\ |(f^{n_i})'(x)| \geq \kappa n_i \qquad \kappa > 0 \end{cases}$$

We can then define a tower map $\mathcal{F} : Y \mapsto Y$, $Y = W \times \mathbb{N}_0$

$$\mathcal{F} : \begin{cases} (x, j) \mapsto (x, j+1), & x \in I_i,\ j < n_i - 1 \\ (x, n_i - 1) \mapsto (f^{n_i}(x), 0) & x \in I_i. \end{cases}$$

Also equip tower with a metric which at the $j$:th level is $m_j = 1/\lambda^j$, $\lambda = e^{-\kappa/2}$. We then obtain

$$\begin{cases} J\mathcal{F}|_{(I_i, j)} = \lambda, & j < n_i - 1 \\ J\mathcal{F}|_{I_i, j} = |(f^{n_i})'|/\lambda^{n_i - 1}, & j = n_i - 1. \end{cases}$$

A *Large deviation estimate* gives

$$\mathrm{Leb}\{x \in W : \tau(x) > n\} \leq Ce^{-\gamma n}, \qquad \gamma > 0$$

The above is sufficient to prove exponential decay of correlation, using that the transfer operator has a spectral gap.
Using Gordin's method one can prove CLT.

## Modification for Hénon maps

We define an equivalence relation on stable manifolds

$$x \sim y \iff d(f^j x, f^j y) \to 0 \qquad \text{as } j \to \infty$$

The base of the tower is now

$$\Lambda = \{x : \pi_1(x) \in W, d(f^j(\pi_1(x), \mathcal{C}) \geq e^{-\alpha j} \ \forall j \geq 1\}$$

Here $\pi_1(x)$ is the projection onto a leave of the unstable manifold of the fixed point along $\mathcal{F}^s$

$$\Lambda = \bigcup \Lambda_i,$$

where $\pi_1(\Lambda_i)$ are Cantor sets of positive one-dimensional measure along $W^u(P_1)$. We will introduce a first return map of equivalence classes

$$\overline{f}^{R_i}(\tilde{\Lambda}_i) \supset \Lambda.$$

Some pieces of $\overline{f}^{R_i}(\tilde{\Lambda}_i)$ will cover $\tilde{\lambda}$ while others will fall in the gaps of $\tilde{\lambda}$ and will be iterated further. Eventually we will have a so that

$$\overline{f}_i^n(\tilde{J}_i) = \tilde{\Lambda},$$

We will also use the notation $\tau(x) = n_i$ for $x \in \Lambda_i$.

We can now carry out similar estimates as in the $1 - d$ case

- Distortion for $J_u$
- Tail estimate $m(\{x \in \tilde{\Lambda} : \tau(x) > n\}) < e^{-\gamma' n}$, $n \geq 1$.

**Theorem**(L.S. Young, Benedicks)

- We have exponential decay of correlation for Hölder observables.

$$|\int \varphi(f^n x)\psi(x)d\mu(x) - (\int \varphi(x)d\mu(x))(\int \psi(x)d\mu(x)| \leq Ce^{-\kappa n}$$

- The partial sums $S_n(x) = \sum_{j=0}^{n-1} \varphi(f^n x)$ satisfies the central limit theorem.

# Hyperbolic inducing Schemes

Let $(X, d)$ be a compact metric space
$f : X \mapsto X$, $f \in C(X)$, $h_{\mathrm{top}}(f) < \infty$

**(I1)** $W \subset X$ is written as

$$W = \bigcup_{J \in S} J$$

Moreover

$$F|_J = f^{\tau(J)}|_J \qquad \text{can be extended to } \overline{J}$$

**(I2)** For every bi-infinite sequence $\underline{a} = (a_n)_{n\in\mathbb{Z}} \in S^{\mathbb{Z}}$ there exists a *unique* sequence $\underline{x} = \underline{x}(\underline{a}) = (x_n = x_n(\underline{a}))_{n\in\mathbb{Z}}$ such that

(1) $x_n \in \overline{J_{a_n}}$ and $f^{\tau(J_{a_n})}(x_n) = x_{n+1}$;

(2) if $x_n(\underline{a}) = x_n(\underline{b})$ for all $n \leqslant 0$ then $\underline{a} = \underline{b}$.

Define the *coding map* by $\pi\colon S^{\mathbb{Z}} \to \cup_{J\in S}\overline{J}$ by

$$\pi(\underline{a}) := x_0(\underline{a}).$$

Denote by $\sigma$ the full left shift on $S^{\mathbb{Z}}$.

The induced map has the following (semi)-conjugacy property:
**Proposition** The coding map $\pi$ on $S^{\mathbb{Z}}$ is a well defined, continuous semi-conjugacy between the (extension of the) induced map $F$ on $\bigcup_{J \in S} \bar{J}$ and the full shift $\sigma$ on $S^{\mathbb{Z}}$:

$$\pi \circ \sigma(\underline{a}) = f^{\tau(J)} \circ \pi(\underline{a})$$

where $J \in S$ is such that $\pi(\underline{a}) \in \bar{J}$. Moreover, the coding $\pi$ is one-to-one on $\check{S}$ with $\pi(\check{S}) = \bigcup_{J \in S} J$.

Proving the existence and uniqueness of equilibrium measures requires some additional condition on the inducing scheme $\{S, \tau\}$:

**(I3)** The set $S^{\mathbb{Z}} \setminus \check{S}$ supports no (ergodic) $\sigma$-invariant measure which gives positive weight to any open subset.

**(I4)** The induced map $F$ has at least one periodic point in $W$.

Condition (I3) is designed to ensure that every Gibbs measure is supported on $\check{S}$ and its projection by $\pi$ is thus supported on $W$ and is $F$-invariant. $\pi$ is one-to one except on the boundary of $J$ and it follows that the boundary (or preimage thereof) supports no measure which gives positive measure to open sets (meant to exclude Gibbs measures).

This projection $\pi$ of the Gibbsmeasure is a natural candidate for the equilibrium measure for $F$. Condition (I4) is used to prove the finiteness of the pressure function.

**Geometric $t$-potential**

The geometric $t$-potential is defined by

$$\varphi_t(x) = -t \log J^u f(x)$$

where $J^U$ is the unstable Jacobian. The case $t = 1$ corresponds to SRB-measures.

**Theorem A** (Pesin, Senti, Zhang, 2016) Let $\{S, \tau\}$ be an inducing scheme of hyperbolic type satisfying Conditions (I3) and (I4). Assume that the potential function $\varphi$ satisfies certain conditions (P1)-(P4). Then

1. there exists a unique equilibrium measure $\mu_\varphi$ for $\varphi$ among all measures in $\mathcal{M}_L(f, Y)$; the measure $\mu_\varphi$ is ergodic;
2. if $\nu_{\varphi^+} = \mathcal{L}^{-1}(\mu_\varphi)$ has exponential tail, then $\mu_\varphi$ has exponential decay of correlations and satisfies the CLT with respect to a class of functions which contains all bounded locally Hölder continuous functions on $Y$.

**Theorem B** (Pesin, Senti, Zhang) Let $f\colon M \to M$ be a $C^{1+\epsilon}$ diffeomorphism of a compact smooth Riemannian manifold $M$ satisfying Conditions (Y0)-(Y5). Then the following statements hold:

1. There exists an equilibrium measure $\mu_1$ for the potential $\varphi_1$ which is a unique SRB measure;
2. Assume that the inducing scheme $\{S, \tau\}$ for $f$ satisfies Conditions (I3) and (I4) and the following condition:
$$S_n = \#\{J \in S\colon \tau(J) = n\} \leqslant e^{hn} \tag{1}$$
with $0 < h < -\int \varphi_1 \, d\mu_1$. Then there exists $t_0 < 0$ such that for every $t_0 < t < 1$ there exists a measure $\mu_t \in \mathcal{M}(f, Y)$ which is a unique equilibrium measure for the potential $\varphi_t$ among measures in $\mathcal{M}_L(f, Y)$, i.e.,
$$h_{\mu_t}(f) + \int_Y \varphi_t \, d\mu_t = \sup_{\mu \in \mathcal{M}_L(f,Y)} \{h_\mu(f) + \int_Y \varphi_t \, d\mu\}.$$

3. If, in addition, the inducing scheme $\{S, \tau\}$ satisfies Condition (L1), then $\mu_t$ is the unique equilibrium measure for $\varphi_t$ among all measures $\mu$ with $h_\mu(f) > h$ i.e.,

$$h_{\mu_t}(f) + \int_Y \varphi_t \, d\mu_t = \sup_{\substack{\mu \in \mathcal{M}(f,Y) \\ h_\mu(f) > h}} \{h_\mu(f) + \int_Y \varphi_t \, d\mu\}.$$

Consider and inducing scheme $\{S, \tau\}$. Let

$$\lambda^- := \inf_{J \in S} \inf_{x \in J} \frac{1}{\tau(J)} \log |J^u F(x)|$$

and

$$\lambda^+ := \sup_{J \in S} \sup_{x \in J} \frac{1}{\tau(J)} \log |J^u F(x)|$$

## Lemma (Senti-B)

*. If Condition (1) is satisfied for a certain h, then the potential function $\varphi_t$ satisfies condition (P4) for all $t^- < t < t^+$ where $t^- := \frac{h + \int_X \varphi_1 \, d\mu_1}{\lambda^- + \int_X \varphi_1 \, d\mu_1}$ and $t^+ := \frac{h + \int_X \varphi_1 \, d\mu_1}{\lambda^+ + \int_X \varphi_1 \, d\mu_1}$*

Senti and I are able to prove a version of (1). Thus we obtain

**Conclusion.**(Senti-B.) Gibbs states exist for $t$ in an open interval $t_0 < t < 1$.