On Mixing for Circular Flows

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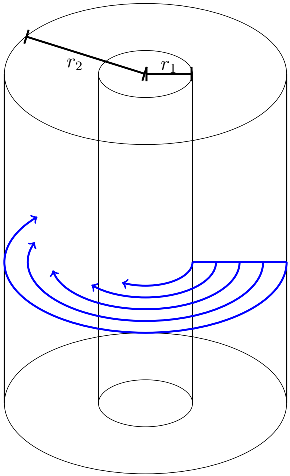
2017-07-03

Circular flows



- 2D Euler equations:

$$\partial_t \omega + v \cdot \nabla \omega = 0,$$
$$v = \nabla^{\perp} \Delta^{-1} \omega.$$

- Stationary solutions:

$$\omega(x, y) = \omega(r),$$
$$v(x, y) = (\partial_r \psi) e_\theta = \begin{pmatrix} -y \\ x \end{pmatrix} \frac{\psi'(r)}{r},$$
$$\psi''(r) + \frac{1}{r}\psi'(r) = \omega(r).$$

Setting and Goals

Linearized Euler equations:

$$\begin{aligned} \partial_t f + U(r)\partial_\theta f &= b(r)\partial_\theta \phi, \\ \Delta_{r,\theta}\phi &= f, \\ \partial_\theta \phi|_{r=r_1,r_2} &= 0, \\ (t,\theta,r) &\in \mathbb{R}\times\mathbb{T}\times[r_1,r_2], \end{aligned}$$

where

$$\begin{aligned} U(r) &= \frac{\psi'(r)}{r}, \\ b(r) &= -\frac{1}{r}\partial_r(\partial_r^2\psi(r) + \frac{1}{r}\partial_r\psi(r)). \end{aligned}$$

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$$W(t,\theta,r) := f(t,\theta - tU(r), r),$$

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$$v = \nabla^\perp \phi \to v_\infty$$

with sharp algebraic decay rates,

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  with sharp algebraic decay rates,
- Explicit boundary layer and blow-up.
- Higher regularity.

Conserved Quanities and Structure

Velocity formulation:

$$\partial_t v + v \cdot \nabla v = \nabla p,$$
$$\nabla \cdot v = 0.$$

- $\|v\|_{L^2}^2 \equiv \text{const.}$

Vorticity formulation:

$$\partial_t \omega + v \cdot \nabla \omega = 0,$$
$$\omega = \nabla \times v,$$
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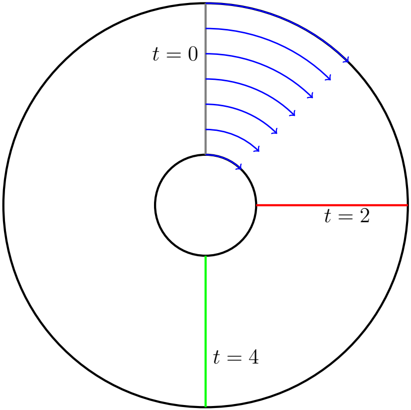
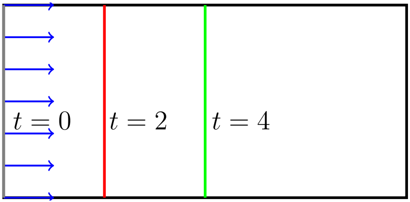
- Hamiltonian system.
- No dissipation, no entropy increase. $\rightsquigarrow$ Damping mechanism?

Taylor-Couette



$$\frac{\psi'(r)}{r} = A + \frac{B}{r^2}, \;\; B = 0.$$



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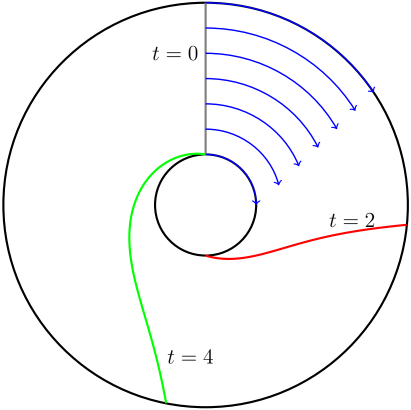

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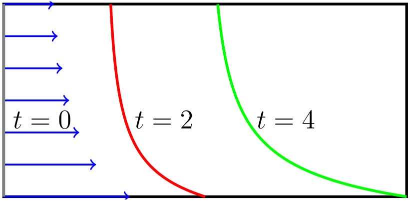
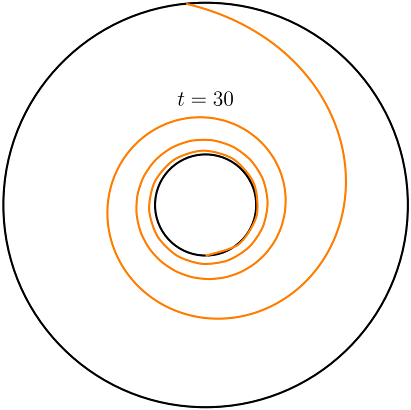


Taylor-Couette: mixing



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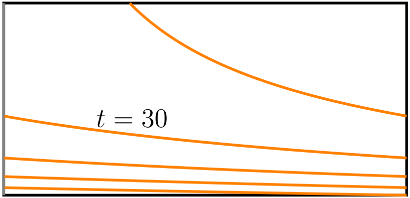
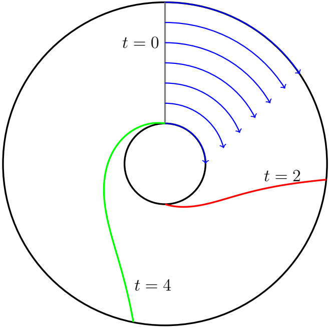
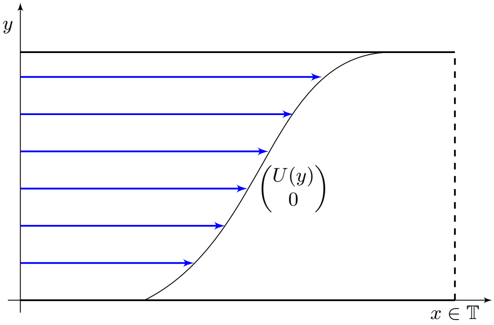


2D Euler equations: periodic channel





$$\partial_t f + U(r)\partial_\theta f = b(r)\partial_\theta \phi,$$
$$(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)\phi = f.$$

$$\partial_t \overline{\omega} + U(y)\partial_x \overline{\omega} = U''(y)\overline{\phi},$$
$$(\partial_x^2 + \partial_y^2)\overline{\phi} = \overline{\omega}.$$

Motivation and related results

- Linear results for Couette, $U(y) = y$, on $\mathbb{T} \times \mathbb{R}$ are classical and explicit

$$\partial_t \overline{\omega} + \begin{pmatrix} y \\ 0 \end{pmatrix} \cdot \nabla \overline{\omega} = 0.$$

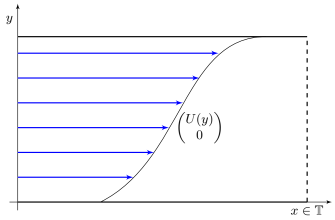
- Nonlinear results of Bedrossian, Germain, Masmoudi, Vicol, Wang on Couette flow on $\mathbb{T} \times \mathbb{R}$ and also for 3D and Navier-Stokes.
- Villani and Mouhot's results on Landau damping

$$\partial_t f + \begin{pmatrix} y \\ F(t,x) \end{pmatrix} \cdot \nabla f = 0,$$

$$\|F(t,x)\| = \mathcal{O}(e^{-\lambda t}).$$

Linear inviscid damping

- $\overline{v} \to (\overline{U}(y), 0)$ as $t \to \infty$.
- Periodic perturbations $x \in \mathbb{T} \rightsquigarrow$ Low-frequency cut-off.
- $y \in \mathbb{R}$: Fourier methods, no boundary conditions [Zil14].
- $y \in [0, 1]$: Boundary effects, blow-up in $H^{3/2+}$ [Zil16a].
- Wei, Zhang, Zhao: Can allow some blow-up (Hardy's inequality); Spectral methods.



Damping & scattering

Theorem (Damping)

For regular, strictly monotone U

$$\|v\|_{L^2_{x,y}} = \mathcal{O}(t^{-1})\|W(t)\|_{H_x^{-1}H_y^{1}},$$

$$\|v_2\|_{L^2_{x,y}} = \mathcal{O}(t^{-2})\|W(t)\|_{H_x^{-1}H_y^{2}},$$

$$\|(y-a)(y-b)\partial_y^2 W(t)\|_{H^{-1}L^2}.$$

Theorem (Scattering)

Suppose $\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-1-\epsilon})$, *then* $\exists W^\infty$:

$$W(t) \xrightarrow[t\to\infty]{L^2} W^\infty.$$

- Core problem: **Regularity/Stability**

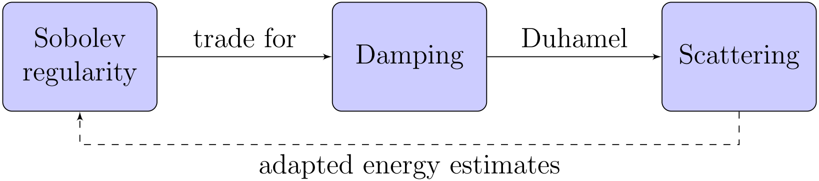
Sketch

$$\|v - \langle v \rangle\|_{L^2} = \|\omega\|_{\dot{H}^{-1}} = \sup_{\phi \in H^1, \|\phi\|_{H^1} \leq 1} \iint \omega \phi$$
$$= \sum_{k \neq 0} \int \hat{\phi} \hat{W} \frac{1}{iktU'} \partial_y e^{iktU}.$$

$$\|v_2\|_{L^2}^2 = \iint \partial_x \omega \Delta^{-1} v_2 = \sum \int ike^{iktU} \hat{W} \mathcal{F}(\Delta^{-1} v_2).$$

$$W(T) = \omega_0 + \int_0^T U'' v_2(t, x - tU(y), y) dy.$$

Strategy


Sobolev regularity
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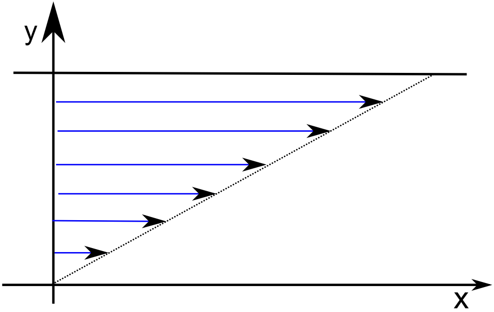
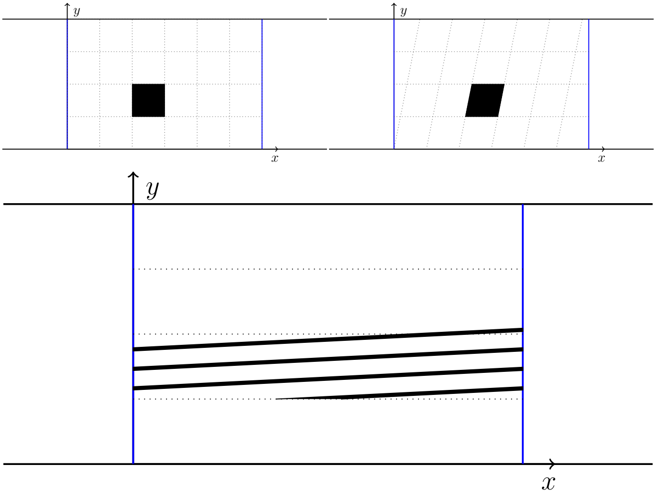
Prototype: linearized Couette flow $U(y) = y$



$$\partial_t \omega + U(y)\partial_x \omega = U''(y) v_2,$$
$$\rightsquigarrow \partial_t \omega + y \partial_x \omega = 0.$$

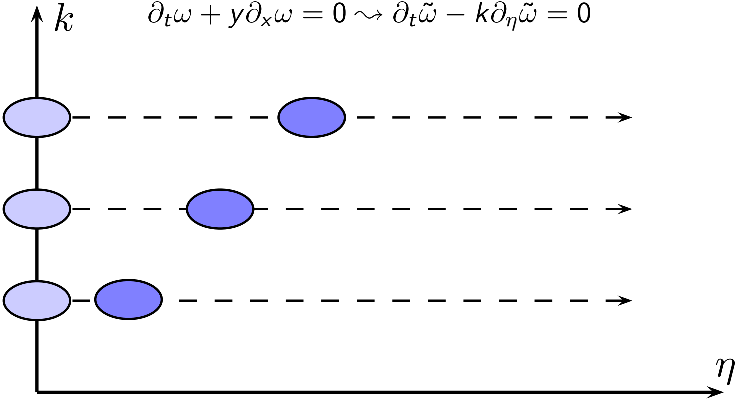
- Free transport.
- Explicitly solvable.

Dynamics



Fourier dynamics

$$\partial_t \omega + y\partial_x \omega = 0 \rightsquigarrow \partial_t \tilde{\omega} - k\partial_\eta \tilde{\omega} = 0$$



Linearized Couette flow

- Explicit solution:

$$\begin{aligned}\omega(t,x,y) =& \omega_0(x-ty,y),\\ \tilde{\omega}(t,k,\eta) =& \tilde{\omega}_0(k,\eta+kt).\end{aligned}$$

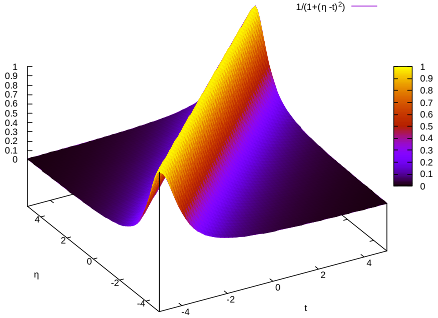
- Velocity field:

$$\vec{v} = \nabla^{\perp}\Delta^{-1}\omega \rightsquigarrow \begin{pmatrix} i\eta \\ -ik \end{pmatrix} \frac{1}{k^2+\eta^2}\tilde{\omega}_0(k,\eta+kt).$$

- Shift in η:

$$\begin{pmatrix} i(\eta-kt) \\ -ik \end{pmatrix} \frac{1}{k^2+(\eta-kt)^2}\tilde{\omega}_0(k,\eta).$$

Fourier multiplier $\frac{1}{k^2+(\eta-kt)^2}$



- Non-uniform decay.
- $\eta \approx kt$ is worst case.
- Penalize with regularity

$$\frac{1}{(k^2+(\eta-kt)^2)(1+\eta^2)}.$$

- Uniform decay.

Model equation

$$\begin{aligned} \partial_t \omega + U(y)\partial_x \omega &= U''\partial_x \phi, \\ \Delta \phi &= \omega. \end{aligned} \quad \text{(LE)}$$

$$\begin{aligned} \partial_t f + y\partial_x f &= c\partial_x \psi, \\ \Delta \psi &= f, \\ c &\in \mathbb{C}. \end{aligned} \quad \text{(CC)}$$

Introduce $\Lambda(t,k,y) = \mathcal{F}_x f(t, \cdot - ty, y)$:

$$\begin{aligned} \partial_t \Lambda &= ikc\Psi, \\ (-k^2 + (\partial_y - ikt)^2)\Psi &= \Lambda. \end{aligned}$$

Explicit solution

$$
\begin{aligned}
\partial_t \Lambda &= ikc\Psi, \\
(-k^2 + (\partial_y - ikt)^2)\Psi &= \Lambda, \\
\rightsquigarrow \partial_t \mathcal{F}\Lambda &= -\frac{ic}{k}\frac{1}{1+(\frac{\eta}{k}-t)^2}\mathcal{F}\Lambda.
\end{aligned}
$$

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$$
\Lambda = \mathcal{F}^{-1}\exp\left(\frac{ic}{k}\int_0^t \frac{1}{1+(\frac{\eta}{k}-\tau)^2}d\tau\right)\mathcal{F}\Lambda_0.
$$

Approaches

- Pseudodifferential or semiclassical calculus.

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- Weighted energy anticipating possible growth.
- (Shifted) elliptic regularity.

Energy estimate

- Decreasing Fourier weight:

$$A(t) : \Lambda \mapsto \mathcal{F}^{-1} \exp\left(\arctan\left(\frac{\eta}{k} - t\right)\right) \mathcal{F}\Lambda.$$

$$\frac{d}{dt}\langle \Lambda, A(t)\Lambda \rangle = - \int \frac{e^{\arctan(\frac{\eta}{k} - t)}}{1 + (\frac{\eta}{k} - t)^2} |\mathcal{F}\Lambda|^2 d\eta + 2\mathrm{Re}\langle \dot{\Lambda}, A\Lambda \rangle.$$

- $\langle \dot{\Lambda}, A\Lambda \rangle = \langle ikc\Psi, A\Lambda \rangle = \langle \frac{ic}{k} \frac{1}{1+(\frac{\eta}{k}-t)^2} \mathcal{F}\Lambda, e^{\arctan(\frac{\eta}{k}-t)} \mathcal{F}\Lambda \rangle.$

L^2 stability

Theorem

Let U be strictly monotone, then for $\|U''\|_{W^{1,\infty}} L$ sufficiently small,

$$\langle W, AW \rangle$$

is non-increasing. In particular,

$$\|W(t)\|_{L^2}^2 \lesssim \langle W, AW \rangle \leq \langle \omega_0, A(0)\omega_0 \rangle \lesssim \|\omega_0\|_{L^2}^2.$$

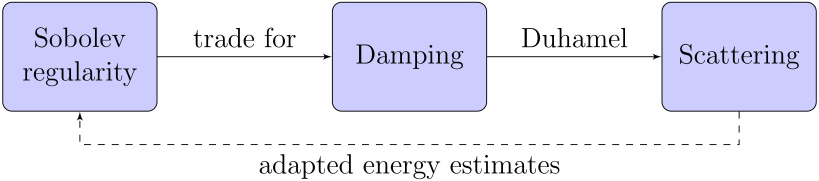
Higher Sobolev norms

$$\begin{aligned} \partial_t \partial_y^j \Lambda &= ick \partial_y^j \Psi, \\ (-k^2 + (\partial_y - ikt)^2) \partial_y^j \Psi &= \partial_y^j \Lambda. \end{aligned}$$

$$\sum_{j' \leq j} \langle \partial_y^{j'} \Lambda, A \partial_y^{j'} \Lambda \rangle \approx \| \Lambda \|_{H^j}^2$$

- Commutator terms in the general case.
- Inductive proof yields stability in any Sobolev space.

Strategy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- Even model problem is non-trivial.
- **Boundary** $\rightsquigarrow$ Corrections to Ψ, blow-up.

Boundary layer

Consider $U(y) = y, \omega_0(x, y) = 2\cos(x)$.

$$\begin{aligned} W(t, 1, y) &\equiv 1, \\ (-1 + (\partial_y - it)^2)\Phi &= 1, \\ \Phi|_{y=0,1} &= 0. \end{aligned}$$

Take a y derivative:

$$\begin{aligned} \partial_y W &\equiv 0, \\ (-1 + (\partial_y - it)^2)\partial_y \Phi &= 0, \\ \partial_y \Phi|_{y=0,1} &= \frac{1}{it} + \mathcal{O}(t^{-2}). \end{aligned}$$

Blow-up

- In general:

$$\partial_y \Phi|_{y=0,1} = \frac{1}{t} \frac{i}{(U')^2}\omega_0\Big|_{y=0,1} + \mathcal{O}(t^{-2})\|W(t)\|_{H^2}.$$

- Evolution of $\partial_y W|_{y=0,1}$:

$$\partial_t \partial_y W|_{y=0,1} = \partial_y(U'' ik\Phi)|_{y=0,1} = U'' ik\partial_y \Phi|_{y=0,1},$$

$$\rightsquigarrow \left|\partial_y W|_{y=0,1}\right| \gtrsim \log(t).$$

Boundary layer

$$
\begin{aligned}
\partial_t W &= b\partial_\theta \Phi = bL_t W, \\
E_t L_t W &= W, \\
L_t W|_{y=a,b} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \partial_y W &= bL_t \partial_y W + b' L_t W + bL_t [E_t, \partial_y] L_t W + H^{(1)}, \\
H^{(1)} &= \partial_y \Phi(a,t) e^{ikt(U(y)-U(a))} u_1 \\
&\quad + \partial_y \Phi(b,t) e^{ikt(U(y)-U(b))} u_2
\end{aligned}
$$

Boundary layer

$$\partial_y \Phi(a,t) = \langle W, e^{ikt(U(y)-U(a))}\tilde{u}_1\rangle = \tilde{L}_t \partial_y W + c\frac{\omega_0(a)}{iktU'(a)}$$

Separate boundary layer

$$\partial_t \beta + bL_t\beta + \tilde{L}_t\beta = c\frac{\omega_0(a)}{iktU'(a)}e^{ikt(U(y)-U(a))}u_1,$$
$$\beta|_{t=0} = 0.$$

The remainder $\partial_y W - \beta$ has higher regularity.

Duhamel

Let $S(t,\tau)$ be the solution map $W(\tau) \mapsto W(t)$,

$$\nu(T) = \int_0^T S(T,t) c \frac{\omega_0(a)}{iktU'(a)} e^{ikt(U(y)-U(a))} u_1 dt.$$

Boundary blow-up at $y = a$.

$$(U(y)-U(a))\nu(T) = \int_0^T c \frac{\omega_0(a)}{iktU'(a)} (U(y)-U(a)) \\ e^{ikt(U(y)-U(a))} S(T-t,0) u_1 dt$$

Summary

- Linear inviscid damping with optimal decay rates holds.
- The boundary layer β is stable in weighted spaces.
- It depends only on the Dirichlet data of ω_0.
- The remainder $\partial_y W - \beta$ is stable in *unweighted* H^2.
- Only need smallness assumption for L^2 estimate.

Christian Zillinger.
Linear inviscid damping for monotone shear flows.
arXiv:1410.7341, to appear in Transactions of the AMS, 2014.

Christian Zillinger.
Linear inviscid damping for monotone shear flows, boundary effects and sharp Sobolev regularity.
PhD thesis, University of Bonn, 2015.

Christian Zillinger.
Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity.
Archive for Rational Mechanics and Analysis, pages 1–61, 2016.

Christian Zillinger.
On circular flows: linear stability and damping.
arXiv preprint arXiv:1605.05959, 2016.