Global Existence of Small Solutions for the Cubic 1D Klein-Gordon Equation

Annalaura Stingo

Université Paris 13

Asymptotic Analysis of Evolution Equations
July 4th, 2017

The problem

We consider the following Cauchy problem:

(KG)

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + u = P(u, \partial_t\partial_x u, \partial_x^2 u; \partial_t u, \partial_x u) & \\ u(t=1) = \varepsilon u_0(x) & t \geq 1, x \in \mathbb{R} \\ \partial_t u(t=1) = \varepsilon u_1(x) & \end{cases}$$

P homogeneous polynomial of degree 3, affine in $(\partial_t\partial_x u, \partial_x^2 u)$ (*quasi-linear problem*); $\varepsilon \ll 1$ small parameter, u_0, u_1 smooth functions, mildly decaying in space ($O(|x|^{-1})$, for $|x| \to +\infty$).

Recall

- Energy of u :
  $E(t, u) := \int \left(|\partial_t u(t, x)|^2 + |\partial_x u(t, x)|^2 + |u(t, x)|^2\right) dx$;
- Linear Dispersive Effect : $\|u(t, \cdot)\|_{L^\infty} \leq C(1 + t)^{-1/2}$.

Global Existence $d \geq 2$:

- $d \geq 3$, Klainerman ('85), Shatah ('85) : (KG) with quadratic nonlinearity, smooth compactly-supported initial data;
- $d = 2$, Ozawa, Tsutaya, Tsutsumi ('96) (*semi-linear case* $P(u, \partial u)$), et ('97) (*quasi-linear case* $P(u, \partial u, \partial^2 u)$);

Results in $d = 1$:

- Moriyama, Tonegawa, Tsutsumi ('97) : maximal time of existence $T_\varepsilon \geq e^{c/\varepsilon^2}$, for a cubic nonlinearity, or a semi-linear one. Exemples of blow-up : Yordanov, Keel-Tao ('99);
- Delort ('01): structure condition on P that ensures the global existence, when initial data are compactly supported;
- Hayashi, Naumkin : ('12) quadratic semi-linear problem.
- Guo, Han, Zhang: ('17) Euler-Poisson system.

Global Existence $d \geq 2$:

- $d \geq 3$, Klainerman ('85), Shatah ('85) : (KG) with quadratic nonlinearity, smooth compactly-supported initial data;
- $d = 2$, Ozawa, Tsutaya, Tsutsumi ('96) (*semi-linear case* $P(u, \partial u)$), et ('97) (*quasi-linear case* $P(u, \partial u, \partial^2 u)$);

Results in $d = 1$:

- Moriyama, Tonegawa, Tsutsumi ('97) : maximal time of existence $T_\varepsilon \geq e^{c/\varepsilon^2}$, for a cubic nonlinearity, or a semi-linear one. Exemples of blow-up : Yordanov, Keel-Tao ('99);
- Delort ('01): structure condition on P that ensures the global existence, when initial data are compactly supported;
- Hayashi, Naumkin : ('12) quadratic semi-linear problem.
- Guo, Han, Zhang: ('17) Euler-Poisson system.

Aim

To prove global existence for (KG) when initial data are not compactly supported, combining the Klainerman vector fields' method with a semiclassical microlocal analysis.

Theorem

Under a structure condition on nonlinearity P (null condition), $\exists s \in \mathbb{N}$ sufficiently large, $\varepsilon_0 \in]0,1]$, such that, for any real initial data $(u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$

$$\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|xu_0\|_{H^1} + \|xu_1\|_{L^2} \leq 1$$

and for any $0 < \varepsilon < \varepsilon_0$, (KG) has a unique solution $u(t,x) \in C^0([1,+\infty[, H^{s+1}(\mathbb{R})) \cap C^1([1,+\infty[; H^s(\mathbb{R}))$. We have the asymptotic development

$$u(t,x) = \Re\left[\frac{\varepsilon}{\sqrt{t}} a_\varepsilon\left(\frac{x}{t}\right) \exp\left[it\varphi\left(\frac{x}{t}\right) + i\varepsilon^2 \left|a_\varepsilon\left(\frac{x}{t}\right)\right|^2 \Phi_1\left(\frac{x}{t}\right)\log t\right]\right] + \frac{\varepsilon}{t^{\frac{1}{2}+\sigma}} r(t,x),$$

with a_ε compactly supported in $[-1,1]$ $\varphi(x) = \sqrt{1-x^2}$, $\Phi_1(x)$ real function obtained from P, and $r(t,x)$ remainder term.

Theorem

Under a structure condition on nonlinearity P (null condition), $\exists s \in \mathbb{N}$ sufficiently large, $\varepsilon_0 \in]0,1]$, such that, for any real initial data $(u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$

$$\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|xu_0\|_{H^1} + \|xu_1\|_{L^2} \leq 1$$

and for any $0 < \varepsilon < \varepsilon_0$, (KG) has a unique solution $u(t,x) \in C^0([1,+\infty[, H^{s+1}(\mathbb{R})) \cap C^1([1,+\infty[; H^s(\mathbb{R}))$. We have the asymptotic development

$$u(t,x) = \Re\left[\frac{\varepsilon}{\sqrt{t}} a_\varepsilon\left(\frac{x}{t}\right) \exp\left[it\varphi\left(\frac{x}{t}\right) + i\varepsilon^2 \left|a_\varepsilon\left(\frac{x}{t}\right)\right|^2 \Phi_1\left(\frac{x}{t}\right) \log t\right]\right] + \frac{\varepsilon}{t^{\frac{1}{2}+\sigma}} r(t,x),$$

with a_ε compactly supported in $[-1,1]$ $\varphi(x) = \sqrt{1-x^2}$, $\Phi_1(x)$ real function obtained from P, and $r(t,x)$ remainder term.

Null condition : Automatically satisfied by Hamiltonian nonlinearities. Examples of nonlinearities that do not satisfy this condition and for which we don't have global existence.

Some ideas of the proof : a toy model

We consider the following model:

$$(KG_{mod}) \quad \begin{cases} \left(D_t - \sqrt{1+D_x^2}\right) u = \alpha |u|^2 u \\ u|_{t=1} = \varepsilon u_0(x) \end{cases} \qquad t \geq 1, x \in \mathbb{R}$$

with $D := \frac{1}{i}\partial$, $u_0(x)$ smooth function, $xu_0(x) \in L^2$, $\alpha \in \mathbb{R}$ (*null condition* on this example).
If we consider the *Klainerman vector field* $Z = t\partial_x + x\partial_t$, then

$$\left(D_t - \sqrt{1+D_x^2}\right) Zu = \alpha |u|^2 (Zu) + \ldots$$

hence the energy inequality :

$$\|Zu(t,\cdot)\|_{L^2} \lesssim \|Zu(1,\cdot)\|_{L^2} + \int_1^t \|u(\tau,\cdot)\|_{L^\infty}^2 \|Zu(\tau,\cdot)\|_{L^2} d\tau$$

Bootstrap Argument

We look for constants $A, B > 0$ sufficiently large, and $\varepsilon_0 > 0$ sufficiently small, such that, $\forall\, 0 < \varepsilon < \varepsilon_0$, if u is solution of (KG_{mod}) in $[1, T]$ and satisfies

$$\text{(1a)} \qquad \|u(t,\cdot)\|_{L^\infty} \leq A\varepsilon t^{-\frac{1}{2}}$$

$$\text{(1b)} \qquad \|u(t,\cdot)\|_{L^2} + \|Zu(t,\cdot)\|_{L^2} \leq B\varepsilon t^{\sigma}$$

for every $t \in [1, T]$, and for a small $\sigma > 0$, then it satisfies also

$$\text{(2a)} \qquad \|u(t,\cdot)\|_{L^\infty} \leq \frac{A}{2}\varepsilon t^{-\frac{1}{2}}$$

$$\text{(2b)} \qquad \|u(t,\cdot)\|_{L^2} + \|Zu(t,\cdot)\|_{L^2} \leq \frac{B}{2}\varepsilon t^{\sigma}$$

Remark : Energy is not uniformly bounded in time.

Bootstrap Argument

We look for constants $A, B > 0$ sufficiently large, and $\varepsilon_0 > 0$ sufficiently small, such that, $\forall\, 0 < \varepsilon < \varepsilon_0$, if u is solution of (KG_{mod}) in $[1, T]$ and satisfies

$$(1a) \qquad \|u(t,\cdot)\|_{L^\infty} \le A\varepsilon t^{-\frac{1}{2}}$$

$$(1b) \qquad \|u(t,\cdot)\|_{L^2} + \|Zu(t,\cdot)\|_{L^2} \le B\varepsilon t^{\sigma}$$

for every $t \in [1, T]$, and for a small $\sigma > 0$, then it satisfies also

$$(2a) \qquad \|u(t,\cdot)\|_{L^\infty} \le \frac{A}{2}\varepsilon t^{-\frac{1}{2}}$$

$$(2b) \qquad \|u(t,\cdot)\|_{L^2} + \|Zu(t,\cdot)\|_{L^2} \le \frac{B}{2}\varepsilon t^{\sigma}$$

Remark : Energy is not uniformly bounded in time.
After the energy inequality for u and Zu, $(1a) + (1b) \Rightarrow (2b)$.

Propagation of the L^∞ estimate

Difficulty : From the *Klein-Gordon* equation (as for the *wave equation*) we are not able to deduce "directly" the wished L^∞ estimates for u. Moreover, a *Klainerman-Sobolev* inequality

$$\|u(t,\cdot)\|_{L^\infty} \leq Ct^{-\frac{1}{2}} E(t, \partial^\alpha Z u)^{\frac{1}{2}}$$

doesn't give the optimal decay $t^{-1/2}$.

In literature : When initial data are compactly supported, the solution remains localised in the light cone (finite speed of propagation) $\Rightarrow$ Hyperbolic coordinates.

New Idea: Deduce from PDE (KG_{mod}) an ODE, using semiclassical pseudo-differential calculus.

Semiclassical Coordinates

We introduce $v(t,x) = \sqrt{t}u(t,tx)$, $h := \frac{1}{t}$ *semiclassical parameter* $(h \to 0)$. Function v is solution of the equation:

$$(KG_{sc}) \qquad D_t v - Op_h^w(\lambda_h(x,\xi))v = h\alpha|v|^2 v$$

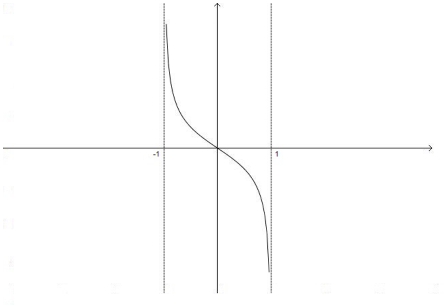
with $\lambda_h(x,\xi) = x\xi + \sqrt{1+\xi^2}$.

Semiclassical Weyl quantization of a symbol $a(x,\xi)$ acting on $w(x)$

$$Op_h^w(a(x,\xi))w(x) := \frac{1}{2\pi h}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a(\frac{x+y}{2},\xi)w(y)\,dyd\xi.$$

We want to develop $\lambda_h(x,\xi)$ in order to transform $Op_h^w(\lambda_h(x,\xi))$ into a product by a real function $\omega(x)$, modulo some integrable remainder.

Let $\Lambda = \{(x,\xi) | \partial_\xi \lambda_h = 0\} = \{(x,\xi) | \xi = d\varphi(x)\}$, $\varphi(x) = \sqrt{1-x^2}$.



Λ for Klein-Gordon equation

Let $\Lambda = \{(x,\xi) | \partial_\xi \lambda_h = 0\} = \{(x,\xi) | \xi = d\varphi(x)\}$, $\varphi(x) = \sqrt{1-x^2}$.

We localize $\lambda_h(x,\xi)$ is a neighbourhood of Λ of size $O(\sqrt{h})$ through an operator $\Gamma = Op_h^w\left(\frac{\partial_\xi \lambda_h}{\sqrt{h}}\right)$ ("wave packets" method by Ifrim-Tataru).

Consequence: $\|\Gamma\|_{\mathcal{L}(L^2;L^\infty)} = O(h^{-1/4})$, better than semiclassical Sobolev injection (loss in $O(h^{-1/2})$).

Let $\Lambda = \{(x,\xi) | \partial_\xi \lambda_h = 0\} = \{(x,\xi) | \xi = d\varphi(x)\}$, $\varphi(x) = \sqrt{1-x^2}$.

We localize $\lambda_h(x,\xi)$ is a neighbourhood of Λ of size $O(\sqrt{h})$ through an operator $\Gamma = Op_h^w\left(\frac{\partial_\xi \lambda_h}{\sqrt{h}}\right)$ ("wave packets" method by Ifrim-Tataru).
Consequence: $\|\Gamma\|_{\mathcal{L}(L^2;L^\infty)} = O(h^{-1/4})$, better than semiclassical Sobolev injection (loss in $O(h^{-1/2})$).

We write $\lambda_h(x,\xi) = \lambda_h(x, d\varphi(x)) + O((\xi - d\varphi(x))^2)$, and hence we deduce $Op_h^w(\lambda_h(x,\xi)) = \underbrace{\lambda_h(x, d\varphi(x))}_{\omega(x)} + O((hD_x - d\varphi)^2)$.

- $(hD_x - d\varphi)v$ can be expressed in terms of hZu;
- $(\xi - d\varphi)^2 = O(\sqrt{h}(\xi - d\varphi))$ on the support of the truncation.

Let $\Lambda = \{(x,\xi) | \partial_\xi \lambda_h = 0\} = \{(x,\xi) | \xi = d\varphi(x)\}$, $\varphi(x) = \sqrt{1-x^2}$.

We localize $\lambda_h(x,\xi)$ is a neighbourhood of Λ of size $O(\sqrt{h})$ through an operator $\Gamma = Op_h^w\big(\frac{\partial_\xi \lambda_h}{\sqrt{h}}\big)$ ("wave packets" method by Ifrim-Tataru).
Consequence: $\|\Gamma\|_{\mathcal{L}(L^2;L^\infty)} = O(h^{-1/4})$, better than semiclassical Sobolev injection (loss in $O(h^{-1/2})$).

We write $\lambda_h(x,\xi) = \lambda_h(x, d\varphi(x)) + O((\xi - d\varphi(x))^2)$, and hence we deduce $Op_h^w(\lambda_h(x,\xi)) = \underbrace{\lambda_h(x, d\varphi(x))}_{\omega(x)} + O((hD_x - d\varphi)^2)$.

- $(hD_x - d\varphi)v$ can be expressed in terms of hZu;
- $(\xi - d\varphi)^2 = O(\sqrt{h}(\xi - d\varphi))$ on the support of the truncation.

Then $\|(hD_x - d\varphi(x))^2 \Gamma v\|_{L^\infty} \lesssim \sqrt{h} h\, h^{-1/4} \|Zu\|_{L^2} = O(h^{5/4-\sigma})$, after (1b).

We obtain

$$\underbrace{D_t v - \omega(x)v - \frac{\alpha}{t}|v|^2 v}_{ODE} = \underbrace{O_{L^\infty}(t^{-5/4+\sigma})}_{\text{integrable remainder}}$$

We deduce an uniform estimate for $\|v(t,\cdot)\|_{L^\infty}$ and its asymptotic behaviour, which gives us the optimal estimate of $\|u(t,\cdot)\|_{L^\infty}$ in $t^{-1/2}$ (hence (2a)), and the asymptotic behaviour of u.

Remark : For a general non-linearity P, other cubic terms appear in the non-linearity $(v^3, |v|^2\bar{v}, \bar{v}^3)$, and they can be eliminated by a normal form argument for ODEs. The *Null Condition* on P is the necessary and sufficient condition for the coefficient α of $|v|^2 v$ to be real.

(A.S.) A. Stingo, *Global existence and asymptotics for quasi-linear one-dimensional Klein-Gordon equations with mildly decaying Cauchy data*, to appear in Bulletin de la SMF.

New Perspectives

We want to study a coupled wave/Klein-Gordon system in $d = 2$:

$$\text{(W-KG)} \quad \begin{cases} \Box u = P(\partial u, \partial v; \partial^2 u, \partial^2 v) \\ \Box v + v = Q(\partial u, \partial v; \partial^2 u, \partial^2 v) \end{cases} \quad t \geq 1, x \in \mathbb{R}^2$$

with small initial data (of size ε), decaying at infinity. P, Q homogeneous polynomials of degree 2.
This system represents a model for the nonlinear interaction between a non-massif field u and a massif field v
The aim :

- To prove the global existence of the solution of (W-KG) ;
- To find general structure conditions on P, Q that ensure the global existence ;
- To adapt our method to this problem.

Thank you for your attention !