Asymptotic behaviour for Landau-Lifshitz and nonlinear heat equations

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Outline: two energy-critical parabolic problems

1. 2D Landau-Lifshitz equations:
   – no blow-up for equivariant solutions of higher degree

2. Energy-critical nonlinear heat equation:
   – global, decaying solutions below threshold

1: 2D Landau-Lifshitz equations

2D Landau-Lifshitz: an energy-critical geometric PDE

- time-dependent maps

$$\vec{u}(\cdot, t) : M \; (= \mathbb{R}^2) \to N \; (= \mathbb{S}^2)$$

i.e. $\vec{u}(x,t) \in \mathbb{R}^3$, $|\vec{u}(x,t)| \equiv 1$ (*magnetization*)

- energy (*exchange*): $\mathcal{E}(\vec{u}(\cdot, t)) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \vec{u}|^2 \, dx$

so $-grad\mathcal{E}(\vec{u}) = Proj_{T_{\vec{u}}\mathbb{S}^2} \Delta \vec{u} = \Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u}$

- some geometric/physical evolution PDE arising from $\mathcal{E}$:

heat-flow (geometry): $\vec{u}_t = \Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u}$

"Schrödinger map": $\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad (= J \; grad\mathcal{E}(\vec{u}))$

Landau-Lifshitz (micromagnetics): $a > 0, \; b \in \mathbb{R}$

$$\boxed{\vec{u}_t = a \left(\Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u} \right) + b \; \vec{u} \times \Delta \vec{u}}$$

Scaling, Criticality, Bubbling, Threshold

In 2 space dimensions, $M = \mathbb{R}^2$, these PDE are energy critical:

$$\vec{u}^{\,s}(x,t) := \vec{u}\left(\frac{x}{s}, \frac{t}{s^2}\right) \implies \mathcal{E}(\vec{u}^{\,s})(t) = \mathcal{E}(\vec{u})(t/s^2), \quad s > 0$$

Question: which initial data lead to globally smooth solutions, and which lead to singularity formation ?

[Struwe 85],[Qing 95],[Harpes 03]...: "Struwe" weak solution has at most finitely many singular points, at which non-constant harmonic maps bubble: eg,

$$\vec{u}(x,t) \approx \vec{H}\left(\frac{x - x_0}{s(t)}\right), \quad s(t) \to 0, \quad \vec{H} : \mathbb{R}^2 \to \mathbb{S}^2 \text{ harmonic}$$

Equivariant Formulation of Landau-Lifshitz

The Landau-Lifshitz equation $\vec{u}_t = a\left(\Delta\vec{u} + |\nabla\vec{u}|^2\vec{u}\right) + b\ \vec{u}\times\Delta\vec{u}$ preserves the symmetry class of *m-equivariant* maps, $m \in \{1, 2, 3, ...\}$:

$$\vec{u}(x,t) = \begin{pmatrix} \sin\phi(r,t)\cos(m\theta + \alpha(r,t)) \\ \sin\phi(r,t)\sin(m\theta + \alpha(r,t)) \\ \cos\phi(r,t) \end{pmatrix}$$

where $(r = |x|, \theta)$ are polar coordinates on $\mathbb{R}^2$.

For the heat-flow $(b = 0)$, the further reduction $\alpha(r,t) \equiv 0$ (*co-rotational*) results in a scalar PDE for $\phi(r,t)$:

$$\boxed{\phi_t = \phi_{rr} + \frac{1}{r}\phi_r - \frac{m^2}{2r^2}\sin(2\phi)}$$

Equivariant Maps: Energy, Topology, BCs, Harmonic Maps

Within the class of *m-equivariant* maps, $m \in \{1, 2, 3, \ldots\}$:

- energy: $\mathcal{E}(\vec{u}) = \pi \int_0^\infty \left\{ \phi_r^2 + \left(\frac{m^2}{r^2} + \alpha_r^2 \right) \sin^2(\phi) \right\} r\, dr$
- choose BC: $\vec{u}|_{r=0} = -\hat{k}, \; \vec{u}|_{r=\infty} = \hat{k} \;\; (\phi|_{r=0} = \pi, \; \phi|_{r=\infty} = 0)$

$$\implies \text{degree}(\vec{u}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \partial_1 \vec{u} \cdot (\vec{u} \times \partial_2 \vec{u}) = m$$

- 'Bogomolnyi' energy lower bound:

  $$\mathcal{E}_{a \le r \le b}(\vec{u}) \ge 2\pi m |\vec{u}_3(b) - \vec{u}_3(a)| = 2\pi m |\cos\phi(b) - \cos\phi(a)|$$

  with equality $\iff$ $\vec{u}$ is harmonic on $[a, b]$: $\exists\, s > 0, \alpha \in \mathbb{R}$,

  $$\vec{u}(x) = \vec{H}_m^{(s,\alpha)} := e^{(m\theta + \alpha)\hat{k}\times} (\sin Q_m(r/s), 0, \cos Q_m(r/s)),$$

  $$Q_m(r) = \pi - 2\tan^{-1}(r^m) \quad (\phi(r) = Q_m\left(\frac{r}{s}\right))$$

  (or a shift or inversion thereof)
- in particular: $\mathcal{E}(\vec{u}) \ge \mathcal{E}(\vec{H}_m) = 4\pi m$ (above threshold)

Slightly Above-Threshold Flow: Harmonic Map 'Stability'

$$\left\{ \begin{array}{c} \vec{u}_t = a\left(\Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u}\right) + b\, \vec{u} \times \Delta \vec{u} \\ m-\text{equivariant}, \ \boxed{0 < \mathcal{E}(\vec{u}) - 4\pi m \ll 1} \end{array} \right\}$$

[G-Nakanishi-Tsai 10]:

- $m \geq 3$: $\mathcal{E}\left(\vec{u}(\cdot,t) - \vec{H}_m^{(s_\infty,\alpha_\infty)}\right) \to 0$ as $t \to \infty$ (scattering sense if $a = 0$).
- $m = 2$, $b = 0$: $\mathcal{E}\left(\vec{u}(\cdot,t) - \vec{H}_2^{(s(t),\alpha(t))}\right) \to 0$ as $t \to \infty$, but $s(t) \to 0$ is possible ('infinite-time singularity')

$m = 1$: finite-time blowup is possible

- heat-flow ($b = 0$): [Chang-Ding-Ye 93], [vdBerg-Hulshof-King 03] (formal asymptotics), [Raphaël-Schweyer 12]
- Schrödinger map ($a = 0$): [Merle-Raphaël-Rodnianski 11], [Perelman 12]

Above-Threshold Heat-Flow: Large Data Global Regularity

Consider now the purely dissipative ($b = 0$), m-equivariant heat-flow, in the co-rotational setting:

$$\begin{cases} \phi_t = \Delta\phi - \frac{m^2}{2r^2}\sin(2\phi) \\ \phi(r,0) = \phi_0(r), \quad \phi(0,t) = \pi, \quad \phi(\infty,t) = 0 \end{cases}$$

Theorem [G. - Roxanas]: If $m \geq 4$ and $\mathcal{E}(\phi_0) \leq 3\mathcal{E}(Q_m)$, the solution is global and smooth with

$$\phi(r,t) \to Q_m(r/s_\infty) \quad (\text{some } s_\infty > 0) \quad \text{as } t \to \infty$$

- main point: for higher-degree maps, there is **no singularity formation**, even though there is sufficient energy
- condition $\mathcal{E}(\phi_0) \leq 3\mathcal{E}(Q_m)$ allows only one bubble
- [Grotowski-Shatah 07]: similar result on a disk, via max. principle

Global Smooth Solution $\leftrightarrow$ No Bubbles

The main point is to exclude (single) bubbling:

$$t_j \to T, \quad \mathcal{E}\big(\phi(\cdot, t_j) - Q_m(\cdot/s_j) - \phi(\cdot, T)\big) \to 0$$
$$\mathcal{E}(\phi(\cdot, t_j)) \to \mathcal{E}(\phi(\cdot, T)) + \mathcal{E}(Q_m)$$

- if $T = \infty$, then $\phi(r, \infty)$ is harmonic, and 'below threshold' $(\mathcal{E}(\phi(\cdot, \infty)) < 2\mathcal{E}(Q_m)$, $\phi(0, \infty) = \phi(\infty, \infty) = 0)$, hence

  $$\phi(r, \infty) \equiv 0, \quad \mathcal{E}(\phi(\cdot, t)) \to \mathcal{E}(Q_m),$$

  and infinite-time blow-up is ruled out by [G.-Nakanishi-Tsai]
- **it remains to rule out finite time blow-up:** $T < \infty$
- can exclude concentration at infinity ($s_j \to \infty$) by (localized) energy dissipation relation: hence $s_j \to 0$.

Key Ingredient: Approximate Solution

At time $t = t_0 < T$ close to the singular time, we have:

$$\phi(r, t_0) = Q_m\left(\frac{r}{s_0}\right) + \phi(r, T) + \xi_0, \qquad s_0 \ll 1, \qquad \mathcal{E}(\xi_0) \ll 1$$

Let $\tilde{\phi}(r, t)$ be the solution for $t \geq t_0$ with data $\tilde{\phi}(r, t_0) = \phi(r, T)$. Since this is 'below-threshold', $\tilde{\phi}$ is global, smooth and decays to 0.

Idea: since $s \ll 1$ and $\tilde{\phi}$ decays,

$$Q_m(r/s) + \tilde{\phi}(r, t) \text{ is a (global) approximate solution.}$$

More precisely, with $Eqn(\phi) := \phi_t - [\Delta\phi + \frac{m^2}{2r^2}\sin(2\phi)]$,

$$\|Eqn[Q_m(\cdot/s) + \tilde{\phi}]\|_{L^2([t_0,\infty);\dot{W}^{1,1})} \to 0 \text{ as } s \to 0.$$

So we may hope to express the (nearby) **true** solution as

$$\phi(r, t) = Q_m(r/s) + \tilde{\phi}(r, t) + \xi(r, t),$$

and control $\xi(r, t)$ beyond the time of singularity.

Key Ingredient: Linearized Evolution and Modulation

The equation for the error has the form:

$$\xi_t + H_s\xi = Eqn[Q_m(\cdot/s) + \tilde{\phi}] + V_s(\tilde{\phi})\xi + \text{nonlinear terms}$$

where the *linearized operator* about harmonic map $Q_m(r/s)$ is

$$H_s = -\Delta + \frac{m^2}{r^2}(1 - 2(h_m^s)^2), \quad h_m^s(r) := \sin Q_m(r/s)$$
$$= (L^s)^* L^s, \qquad L^s = h_m^s \partial_r \frac{1}{h_m^s} = \partial_r - \frac{(h_m^s)_r}{h_m^s}$$

Note $h_m^s \in \ker H_s$ (scale invariance), so linearized solutions do not decay. We must modulate the scale, $s = s(t)$, to impose $\xi \perp h_m^s$:

$$\xi_t + H_{s(t)}\xi = Eqn[Q_m(\cdot/s) + \tilde{\phi}] - m\frac{\dot{s}}{s}h\left(\frac{r}{s}\right) + V_s(\tilde{\phi})\xi + \text{nonlinear terms}.$$

Key Ingredient: Linearized Decay Estimates

Linearized problem:

$$\xi_t + (L^s)^* L^s \xi = F, \qquad \xi \perp h_m^s \in \ker L^s$$

[G.-Nakanishi-Tsai]: apply L^s: $\eta := L^s \xi$,

$$\eta_t + L^s (L^s)^* \eta = L^s F,$$

Now $L^s (L^s)^* > -\Delta + \frac{1}{r^2}$, and so heat-equation estimates hold, eg:

$$\|\eta\|_{L_t^\infty L_r^2 \cap L_t^2 L_r^\infty} \leq \|\eta_0\|_{L^2} + \|L^s F\|_{L_t^1 L_r^2 + L_t^2 L_r^1}.$$

Finally, recover ξ from $\eta = L^s \xi$ by solving an ODE, eg:

$$\xi \perp h_m^s \quad \Longrightarrow \quad \|\xi_r\|_{L^p} + \|\frac{\xi}{r}\|_{L^p} \lesssim \|\eta\|_{L^p}.$$

Complete the Argument

$$\phi(r,t) = Q_m(r/s(t)) + \tilde{\phi}(r,t) + \xi(r,t), \quad \xi \perp h_m^s,$$

$$\begin{aligned} \xi_t + H_{s(t)}\xi = Eqn[Q_m(\cdot/s) + \tilde{\phi}] - m\frac{\dot{s}}{s}h\left(\frac{r}{s}\right) \\ + V_s(\tilde{\phi})\xi \\ + \text{nonlinear terms.} \end{aligned}$$

Using that $Q_m(\cdot/s) + \tilde{\phi}$ is an **approximate solution** (as above), along with the **linearized decay estimates**, we find

$$\|\xi_r\|_{L_t^\infty L_r^2 \cap L_t^2 L_r^\infty [t_0,T)} \lesssim \mathcal{E}(\xi_0) \ll 1,$$

$$\|\log\left(\frac{s}{s(0)}\right) - 1\|_{L_t^\infty [t_0,T]} \lesssim (\mathcal{E}(\xi_0))^2 \ll 1$$

for t_0 sufficiently close to T, contradicting the bubbling.

Remark: alternate approach to estimates for heat-flow

- $\phi_t = \Delta\phi - \frac{m^2}{2r^2}\sin(2\phi) = (\partial_r + \frac{1}{r} - \frac{m}{r}\cos(\phi))(\phi_r + \frac{m}{r}\sin(\phi))$
- $\implies \quad q := \phi_r + \frac{m}{r}\sin(\phi) \quad \boxed{q_t + H_\phi q = \frac{m}{r}\sin(\phi) q^2}$
  $H_\phi = -\Delta + \frac{(m-1)^2}{r^2} + \frac{2m}{r^2}(1-\cos(\phi)) \geq -\Delta + \frac{(m-1)^2}{r^2}$
- for $\phi = Q(\cdot/s) + \tilde{\phi} + \xi$,
  $\widehat{q} := q - \tilde{q}, \quad \tilde{q} := (Q(\cdot/s) + \tilde{\phi})_r + \frac{m}{r}\sin(Q(\cdot/s) + \tilde{\phi})$

  $$\|\widehat{q}(t_0)\|_{L^2} \ll 1, \quad \widehat{q}_t + H_\phi \widehat{q} = \text{“}\, \frac{\xi^2\tilde{q}}{r^2} + \frac{1}{r}\sin(Q+\tilde{\phi})(\widehat{q}^2 + \tilde{q}\widehat{q}) + \cdots \text{”}$$

- can estimate $\|\widehat{q}\|_{L^\infty_t L^2 \cap L^2_t L^\infty} \ll 1$, and then recover estimates for ξ by $\widehat{q} \approx L^s \xi$ as above

Remark: extension to equivariant Landau-Lifshitz

For equivariant maps $\vec{u}(x,t) = e^{m\theta R}\vec{v}(r,t)$, $\quad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

the Landau-Lifshitz equation reads

$$\vec{v}_t = \left(aP^{\vec{v}} + b\vec{v}\times\right)\left(\Delta + \frac{m^2}{r^2}R^2\right)\vec{v}.$$

Generalized Hasimoto transform [Chang-Shatah-Uhlenbeck 00]:
$\vec{v}_r - \frac{m}{r}P^{\vec{v}}\widehat{k} = q_1\widehat{e} + q_2(\vec{v}\times\widehat{e})$, $\quad D_r^{\vec{v}}\widehat{e} \equiv 0$. For $q(r,t) = q_1 + iq_2$:

$$\boxed{q_t + (a+ib)LL^*q = -iSq} \quad L = \partial_r + \frac{m}{r}v_3,$$

and $S_r = Re\left(\overline{q} + \frac{m}{r}P^{\vec{v}}\widehat{k}\cdot(\widehat{e} - i\vec{v}\times\widehat{e})\right)(ia-b)L^*q$.

By working with this equation for q as in the previous slide, we expect to prove:

Conjecture: there is no (single) bubbling in the $m \geq 4$ equivariant Landau-Lifshitz flow.

2: Energy-critical nonlinear heat equation

Energy-critical NLH

$$\boxed{u_t = \Delta u + u^3, \quad u(x,0) = u_0(x) \in \dot{H}^1(\mathbb{R}^4)}$$

- energy dissipation: $E(u) = \int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\right)$
  $$E(u(t)) + \int_0^t \int u_t^2 \, dx \, ds = E(u_0)$$
- critical scaling: $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad E(u_\lambda) = E(u)$
- L^2-relation: $\frac{d}{dt}\frac{1}{2}\int u^2 = -\int \left(|\nabla u|^2 - u^4\right) =: -K(u)$
- static solutions: $W(x) = (1 + \frac{|x|^2}{8})^{-1}, \quad \Delta W + W^3 = 0$
- sharp Sobolev: $\int u^4 \leq \left(\int W^4\right)^{-1} \left(\int |\nabla u|^2\right)^2$

Global, decaying solutions below threshold

$$u_t = \Delta u + u^3, \quad u(x,0) = u_0(x) \in \dot{H}^1(\mathbb{R}^4)$$

Theorem: $E(u_0) \leq E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2} \implies$
$\exists\,!$ global, smooth solution with $\|\nabla u(t)\|_{L^2} \to 0,\ t \to \infty$

- energy diss. + Sobolev $\implies$ $\sup_t \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2}$
-
  - $u \equiv W$ non-decaying solution
  - $u_0 \in H^1$, $E(u_0) \leq E(W)$, $\|\nabla u_0\|_2 > \|\nabla W\|_2 \implies$ blow-up (variant of [Levine 73])
  - [Schweyer 12]: $E(u_0) = E(W) + \varepsilon$ blow-up constructions
- c.f. recent work (blow-up/classification): Cortazer-delPino-Musso, delPino-Musso-Wei, Collot-Merle-Raphaël, Matano-Merle
- motivation: apply non-classically parabolic methods as in [Kenig-Merle 06] for NLS, [Kenig-Koch 11] for N-S

Local existence theory

Step 1: Local theory for

$$\boxed{u_t = \Delta u + u^3, \quad u(x,0) = u_0(x) \in \dot{H}^1(\mathbb{R}^4)}$$

via standard spacetime estimates for $e^{t\Delta}$ and fixed-point argument:

$\exists\,!$ (maximal-lifespan) solution

$$u \in C_t \dot{H}^1 \cap L^6_{x,t} \cap L^2_t \dot{H}^2(\mathbb{R}^4 \times [0, T_{max}(u_0)))$$

with, eg,

- $T_{max} < \infty \implies \|u\|_{L^6_{x,t}(\mathbb{R}^4 \times [0,T_{max}))} = \infty$
- $\|\nabla u_0\|_2 \le \varepsilon_0 \implies T_{max} = \infty$,

  $u \in L^6_{x,t}([0,\infty)), \quad \nabla u \in L^3_{x,t}([0,\infty))$

Decay of below-threshold global solutions

Step 2: Global solutions decay (as [Gallagher-Iftimie-Planchon 02] for NS)

$T_{max} = \infty$ and $\sup_t \|\nabla u(t)\|_2 < \|\nabla W\|_2$
$\implies u \in L^6_{x,t}([0,\infty))$ and $\|\nabla u(t)\|_2 \to 0$

1. $u \in L^6_{x,t}[0,\infty) \implies \|\nabla u(t)\|_2 \to 0$ (split Duhamel integral)
2. if $u_0 \in L^2$, $\frac{d}{dt}\int u^2 = -2K(u) \lesssim -\int |\nabla u|^2 \implies \nabla u \in L^2_{x,t}$
   $\implies \exists \bar{t}\ \|\nabla u(\bar{t})\|_2 < \varepsilon_0 \implies u \in L^6_{x,t}([0,\infty))$ (small-data theory)
3.
   - $u_0 = w_0 + v_0$, $\|\nabla w_0\|_2 \ll 1$, $v_0 \in H^1$
   - $w \in L^6_{x,t}[0,\infty)$ solution from small data w_0
   - same L^2 argument for $v(t) = u - w \implies v \in L^6_{x,t}([0,\infty))$

Existence of critical element

Step 3: Minimal blow-up solution

$\exists\ E_c \in (0, \|\nabla W\|_2]$ maximal s.t.

$\sup_{t\in[0,T_{max})} \|\nabla u(t)\|_2^2 < E_c \implies T_{max} = \infty, u \in L^6_{x,t}([0,\infty))$

$$\exists \text{ sol. } u_c, \quad \sup_{t\in[0,T_{max})} \|\nabla u_c(t)\|_2^2 = E_c, \quad \|u_c\|_{L^6_{x,t}([0,T_{max})} = \infty,$$
$$\left\{ \frac{1}{\lambda(t)} u_c\left(\frac{x-X(t)}{\lambda(t)}, t\right) \mid t \in [0, T_{max}) \right\} \ \dot{H}^1\text{-precompact}$$

- proof follows [Kenig-Merle 06], [Killip-Visan 10], based on profile decomposition ([Bahouri-Gérard 99],[Keraani 01]...) associated to $e^{t\Delta}$
- goal: $E_c = \|\nabla W\|_2^2$
- if $E_c < \|\nabla W\|_2^2$, then $T = T_{max}(u_c) < \infty$ (and hence $\lambda(t) \to \infty$, $t \to T$), and so it remains to **exclude compact, finite-time blow-up**.

Bounded centre of compactness

Step 4: $|X(t)| \not\to \infty$ (by cut-off energy dissipation relation)

- $\underline{\mathrm{E}} := \inf_{t<T} E(u_c(t)) > 0$ (via (cut-off) L^2-relation, $K(u) = 2E(u) - \frac{1}{2}\int u^4$)
- $t_0 < T,\ e(t_0) := \int \left(\frac{1}{2}|\nabla u_c(t_0)|^2 - \frac{1}{4}u_c(t_0)^4\right)\chi_{|x|\geq R_0} \leq \frac{1}{4}\underline{\mathrm{E}}$
- if $t_n \to T$, $|x(t_n)| \to \infty$, then $e(t_1) \geq \frac{3}{4}\underline{\mathrm{E}}$, some $t_1 \in (t_0, T)$
- $\frac{d}{dt}e(t) = -\int((u_c)_t)^2\chi - \int(u_c)_t\nabla u_c \cdot \nabla\chi \lesssim \|\nabla u_c\|_2\|(u_c)_t\|_2$
- by compactness and energy-dissipation, $0 < \frac{1}{2}\underline{\mathrm{E}} \leq \int_{t_0}^{t_1} \frac{d}{dt}e(t) \lesssim \|\nabla u_c\|_{L_t^\infty L^2}\|(u_c)_t\|_{L_t^2 L^2}\sqrt{T - t_0} \to 0$ as $t_0 \to T$

Step 5: local small energy regularity:

$$\varepsilon := \|u\|_{L_t^\infty(\dot{H}^1 \cap L^4)(B_1 \times (-1,0))} \leq \varepsilon_0 \implies \max_{\overline{B}_{\frac{1}{2}} \times [-\frac{1}{2},0]} |D^k u| \leq c_k \varepsilon$$

(can prove via successive cut-offs and energy estimates).

Step 6: backward uniqueness and unique continuation:

as in [Escuariaza-Seregin-Sverak 02] for N-S

to conclude that since u_c is regular and $\to 0$ away from the origin as $t \to T$, we must have $u_c \equiv 0$.