Various aspects of the dynamic of the cubic Szegő equation

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based on jointwork with Patrick Gérard (Orsay),
numerical simulations by Erwan Faou (INRIA Rennes)

A baby example

Recall

$$\mathcal{C}(e^{ix}) = \left(\frac{s_j e^{i\Psi_j} - e^{ix} \tilde{s}_k e^{i\tilde{\Psi}_k}}{s_j^2 - \tilde{s}_k^2} \right)_{1 \le j,k \le N}.$$

and

$$u(e^{ix}) = \langle \mathcal{C}(e^{ix})^{-1}(1_N), 1_N \rangle.$$

Still valid if $\tilde{s}_N = 0$.

A baby example

$$s_1 = 1 + \varepsilon\ ,\ \tilde{s}_1 = 1\ ,\ s_2 = 1 - \varepsilon$$

State 1: $\psi_1 = 0\ ,\ \tilde{\psi}_1 = 0\ ,\ \psi_2 = \pi$

$$\left\langle \begin{pmatrix} \frac{1+\varepsilon-e^{ix}}{(1+\varepsilon)^2-1} & \frac{1}{1+\varepsilon} \\ \frac{-(1-\varepsilon)-e^{ix}}{(1-\varepsilon)^2-1} & \frac{-1}{1-\varepsilon} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2 \times \mathbb{C}^2} = \frac{2e^{ix}(1-\varepsilon^2)+3\varepsilon}{2-\varepsilon e^{ix}}$$

State 2: $\psi_1 = 0\ ,\ \tilde{\psi}_1 = 0\ ,\ \psi_2 = 0$

$$\left\langle \begin{pmatrix} \frac{1+\varepsilon-e^{ix}}{(1+\varepsilon)^2-1} & \frac{1}{1+\varepsilon} \\ \frac{(1-\varepsilon)-e^{ix}}{(1-\varepsilon)^2-1} & \frac{1}{1-\varepsilon} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2 \times \mathbb{C}^2} = \frac{2+\varepsilon^2-2e^{ix}(1-\varepsilon^2)}{2-(2-\varepsilon^2)e^{ix}}$$

A cousin of the baby example

Start with the datum

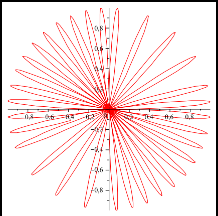
$$u_0^{\varepsilon}(x) = \mathrm{e}^{ix} + \varepsilon\ .$$

Then (movie Erwan Faou)

The trajectory of $p^\varepsilon(t)$ and consequences

$$u^\varepsilon(t,x) = \frac{a^\varepsilon(t)\mathrm{e}^{ix} + b^\varepsilon(t)}{1 - p^\varepsilon(t)\mathrm{e}^{ix}}$$

$$p^\varepsilon(0) = 0\ ,\ 1 - |p^\varepsilon(t_n^\varepsilon)| \sim \varepsilon^2\ ,\ t_n^\varepsilon \sim \frac{(2n+1)\pi}{2\varepsilon}$$



The trajectory of $p^\varepsilon(t)$ and consequences

$$\|u^\varepsilon(t_n^\varepsilon)\|_{H^1}^2 \sim \int_0^{2\pi} |\partial_x u^\varepsilon(t_n^\varepsilon, x)|^2 \, dx \sim \frac{1}{\varepsilon^2} \sim |t_n^\varepsilon|^2$$

$$\lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_0^{2\pi} |\partial_x u^\varepsilon(t, x)|^2 \, dx \right)^{1/2} dt = \infty.$$