Thinning selfdecomposable point processes

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Framework

Let X be a LCSM 'phase' space, $\Phi, \Psi, \dots$ - point processes (PPs) on X (PP is a measurable mappling from $(\Omega, \mathcal{F}, \mathbf{P})$ into the set $\mathcal{N}$ of locally-finite counting measures on X blah-blah-blah...)
The *probability generating functional* p.g.fl. characterises the distribution of a PP Φ:

$$G_\Phi[h] = \mathbf{E}\exp\Big\{\int_X \log h(x)\,\Phi(dx)\Big\} = \mathbf{E}\prod_{x_i\in\Phi} h(x_i)$$

for the class $\mathcal{V}$ of functions $h:\ X \mapsto [0,1]$ such that $\operatorname{supp}(1-h)$ is bounded ($\log 0 = -\infty$ by convention).

Independent thinning

Given a $\varphi \in \mathcal{N}$ and $p \in [0,1]$, denote by $p \circ \varphi$ the independent thinning when each point of φ is retained with probability p and removed with prob. $1-p$ independently of the other points; $p \circ \Phi = p \circ \Phi(\omega)$.
It is easy to see that

$$G_{p\circ\Phi}[h] = G_\Phi[1-p+ph],\ p \in \mathcal{V}.$$

The (stochastic) operation $\circ$ is associative and distributive w.r.t. superposition of PPs:

$$p_1 \circ (p_2 \circ \Phi = (p_1p_2)\circ\Phi,\ p\circ(\Phi_1+\Phi_2) = p\circ\Phi_1 + p\circ\Phi_2.$$

Thinning-stable PPs

Definition

A PP Φ (or its distribution) is called discrete (or thinning) α-stable (notation: DαS) if for any $p \in [0,1]$

$$p^{1/\alpha}\circ\Phi' + (1-p)^{1/\alpha}\circ\Phi'' \overset{\mathcal{D}}{=} \Phi$$

where Φ' and Φ'' are independent copies of Φ.

DαSPPs are fully characterised in Yu. Davydov, I. Molchanov & Z.'11.

CLT for superposition

Let $\Psi_1, \Psi_2, \dots$ be a sequence of i.i.d. PPs. If there exists a PP Φ such that for some α we have

$$n^{-1/\alpha} \circ (\Psi_1 + \dots \Psi_n) \Longrightarrow \Phi \quad \text{as } n \to \infty$$

then Φ is DαS.

Idea

Take $0 < p < 1$ and decompose $S_n = \sum_{i=1}^n \Psi_i \overset{\mathcal{D}}{=} S_{pn} + S_{(1-p)n}$. Then

$$n^{-1/\alpha} \circ S_n \overset{\mathcal{D}}{=} p^{1/\alpha} \circ [(pn)^{-1/\alpha} \circ S_{pn}] + (1-p)^{1/\alpha} \circ [((1-p)n)^{-1/\alpha} \circ S_{(1-p)n}]$$
$$\Longrightarrow p^{1/\alpha} \circ \Phi' + (1-p)^{1/\alpha} \circ \Phi''.$$

NB. The case $\alpha = 1$ corresponds to the classical Poisson limit theorem: Φ is Poisson PP.

Not equally distributed summands

When Ψ_i's are independent, but not i.i.d. , we can still decompose S_n into two independent terms:

$$n^{-1/\alpha} \circ S_n = p^{1/\alpha} \circ [(pn)^{-1/\alpha} \circ S_{pn}] + n^{-1/\alpha} \circ \sum_{j=pn+1}^{n} \Psi_j$$
$$\Longrightarrow p^{1/\alpha} \circ \Phi' + \Phi(p)$$

for some PP $\Phi(p)$ independent of Φ.

Selfdecomposable PPs

It will be convenient to set $p = e^{-t}, \ t \geq 0$ to have an additive semigroup:

$$e^{-t_1} \circ e^{-t_2} \circ \Phi = e^{-(t_1+t_2)} \circ \Phi; \ e^0 \circ \Phi = \Phi.$$

Definition

A PP Φ (or its distribution) is called selfdecomposable (notation: SD) if for any $t > 0$ there exists a PP Φ_t independent of Φ such that

$$\Phi \overset{\mathcal{D}}{=} e^{-t} \circ \Phi' + \Phi_t,$$

where Φ' is an independent copy of Φ.

Thus if $n^{-1/\alpha} \circ S_n \Rightarrow \Phi$, then Φ is necessarily SD.

Infinite divisibility

All possible limits of the sums in the triangular array constitute the class of infinitely divisible distributions:

Definition

A PP Φ (or its distribution) is called infinitely divisible (notation: ID) if for any natural n there exists a PP $\Phi^{(n)}$ independent of Φ such that

$$\Phi \overset{\mathcal{D}}{=} \Phi_1^{(n)} + \dots + \Phi_n^{(n)},$$

where $\Phi_i^{(n)}$'s are independent copies of $\Phi^{(n)}$.

Obviously, we have

$$\text{D}\alpha\text{S} \subset \text{SD} \subset \text{ID}.$$

Integer random variables

Discrete stability and selfdecomposability was introduced by Steutel & van Harn for r.v.'s in $X = \mathbb{Z}_+$ who defined a stochastic 'discrete multiplication' as follows:

$$t \circ \xi \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\xi} \beta_i ,$$

where $\{\beta_i\}$ are independent Bern(t) r.v.'s, and characterised the corresponding discrete-stable and selfdecomposable integer random variables.

Selfdecomposable integer random variables

Theorem (Steutel & van Harn '79)

A non-negative integer valued r.v. ξ is SD iff there exists an ID r.v. $\tilde{\xi}$ with p.g.f. $\tilde{g}$ and $\mathbf{E}\log(1+\tilde{\xi}) < \infty$ such that the p.g.f. of ξ has the form

$$g(z) = \exp\left\{\int_z^1 \frac{\log \tilde{g}(x)}{1-x} dx\right\},\ 0 \le z \le 1. \qquad (1)$$

Remark

A non-negative integer r.v. can be viewed as a PP on X being a singleton and the discrite multiplication corresponds to the independent thinning. We aim to generalise (1) to PPs.

Poisson cluster PPs

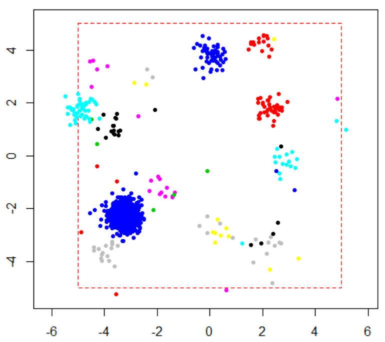

Figure: Centres form a Poisson PPs, each centre gives rise to independent 'daughter' PPs – clusters.

The centres need not live on the same phase space X: given some measure space $[Y, \mu]$ and a family of PP distributions indexed by $y \in Y$ with p.g.fl.'s $\tilde{G}[h|y]$, the Poisson cluster process has p.g.fl.

$$G[h] = \exp\left\{\int_Y (\tilde{G}[h|y] - 1)\,\mu(dy)\right\}.$$

Theorem (Kerstan & Matthes '78)

A finite PP Φ is ID iff there exists a finite PP N with $\mathbf{P}\{N(X) = 0\} = 0$ and $\gamma > 0$ such that $\Phi \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\nu} N_i$, where $\nu \sim$ Po(γ) and $N_i \sim N$ i.i.d. Equivalently,

$$\log G_\Phi[h] = \gamma(G_N[h] - 1)$$

so that Φ is a Poisson cluster process (Y is a singleton and $\mu(Y) = \gamma$).

Finite SD point processes

Main theorem

A finite PP Φ is SD iff there exists a finite PP $\tilde N$ satisfying $\mathbf{E}\log(1+\tilde N(X))<\infty$ and $\gamma>0$ such that $\Phi \overset{\mathcal{D}}{=} \sum_{s_i\in\Pi_\gamma} e^{-s_i}\circ \tilde N_i$, where Π_γ is a Poisson PP on $\mathbb{R}_+$ with density γ and $\tilde N_i\sim\tilde N$ i.i.d. Equivalently,

$$\log G_\Phi[h]=\gamma\int_0^\infty (G_{\tilde N}[1-e^{-s}+e^{-s}h]-1)\,ds \tag{2}$$

so that Φ is a Poisson cluster process with clusters $e^{-t}\circ\tilde N,\ t\in\mathbb{R}_+$.

Corrolary

When X is a singleton, $z=h$, $x=1-e^{-s}+e^{-s}h$ and ID $\tilde\xi=\sum_{i=1}^{\nu}\tilde N_i,\ \nu\sim$ Po(γ), we get the Steutel & van Harn characterisation of integer SD r.v.'s.

Limit theorems for thinned point processes
Integer r.v.'s
Finite SD PPs
General branching operation

Sufficiency

If $\Phi \overset{\mathcal{D}}{=} \sum_{s_i\in\Pi_\gamma} e^{-s_i}\circ\tilde N_i$, then

$$e^{-t}\circ\Phi=\sum_{s_i\in\Pi_\gamma} e^{-(s_i+t)}\circ\tilde N_i \overset{\mathcal{D}}{=} \sum_{s_i\in\Pi_\gamma|_{[t,\infty)}} e^{-s_i}\circ\tilde N_i.$$

Hence $\Phi \overset{\mathcal{D}}{=} e^{-t}\circ\Phi'+\Phi_t$, where

$$\Phi_t \overset{\mathcal{D}}{=} \sum_{s_i\in\Pi_\gamma|_{[0,t)}} e^{-s_i}\circ\tilde N_i$$

independent of Φ and Φ'.

NB

Notice that $\Phi(B)$ is an SD integer r.v. so that $\log(1+\tilde N(B))<\infty$ for any bounded measurable $B\subset X$ guarantees a.s. local finiteness of Φ.

Limit theorems for thinned point processes
Integer r.v.'s
Finite SD PPs
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Necessity: idea of the proof

If Φ is a finite SD PP, then Φ is ID, so that for any $t>0$ and $h\in\mathcal{V}$,

$$\begin{aligned}\log G_{\Phi_t}[h]&=\log G_\Phi[h]-\log G_\Phi[1-e^{-t}+e^{-t}h]\\&=\gamma(G_N[h]-G_N[1-e^{-t}+e^{-t}h])\\&=-\gamma\int_0^t \frac{d}{ds}G_N[1-e^{-s}+e^{-t}h]\,ds.\end{aligned}$$

Thus Φ_t for any t is also ID and its Khinchine measures K_n^t are all non-negative:

$$\log G_{\Phi_t}[h]=K_0^t+\sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} h^{\otimes n}dK_n^t.$$

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By direct computation, denoting $h_s=1-e^{-s}+e^{-t}h$,

$$\frac{d}{ds}G_N[h_s]=L_N(h_s)[1-h_s],\ \text{where}$$

$$L_N(h)[g]=\mathbf{E}\sum_{x_i\in N} g(x_i)\prod_{x_j\neq x_i} h(x_j).\ textTherefore$$

$$\begin{aligned}\lim_{t\downarrow 0} t^{-1}\log G_{\Phi_t}[h]&=-\gamma L_N(h)[1-h]\\&=\gamma\,\mathbf{E}\Big[N(X)\prod_{x_j\in N}h(x_j)-\sum_{x_i\in N}\prod_{x_j\neq x_i}h(x_j)\Big]=1-J_0\end{aligned}$$

$$+\sum_{n=1}^\infty\frac{1}{n!}\int_{X^n}\prod_{i=1}^n h(x_i)\{nJ_n(dx_1\times\cdots\times dx_n)-J_{n+1}(dx_1\times\cdots\times dx_n\times X)\},$$

where J_n are the Janossy measures for N.

As the limit of non-negative Khinchine measures, $\tilde{J}_0 = 1 - J_0 \geq 0$ and

$$\tilde{J}_n(B) = nJ_n(B) - J_{n+1}(B \times X) \geq 0 \ \forall B \subset X^n$$

and magically $\sum_{n=0}^{\infty} \frac{1}{n!}\tilde{J}_n(X^n) = 1$ so that they are Janossy measures for some PP $\tilde{N}$ and

$$\log G_{\Phi_t}[h] = \gamma \int_0^t (G_{\tilde{N}}[h_s] - 1)\, ds$$

implying $\log G_{\Phi}[h] = \gamma \int_0^{\infty} (G_{\tilde{N}}[h_s] - 1)\, ds$.

Extensions and open problems

- The Sufficiency proof applies for a non-finite $\tilde{N}$. Neccesity is yet to be established (working with KLM measures instead)
- If Φ is DαS, $L[v] = G_{\Phi}[1 - v]$ is a Laplce functional of an α-stable random measure μ and Φ is a Cox process directed by μ. For SD Φ, $L[v]$ satisfies the SD equation. Is L still corresponds to a SD measure and how is it related to Φ?
- How $\tilde{N}$ shows in the conditions for the CLT to hold?

General branching operation

The thinning is a particular case of a general associative and distributive operation $\bullet$ – subcritical branching operation. Limit theorems, DαSand SD can be considered then w.r.t. $\bullet$.

Requirements on the operation

$\forall\, t, s \in (0, 1]$ and $\forall\, \varphi, \varphi_1, \varphi_2$ finite counting measures on X

1. Associativity with respect to superposition:
$$t \bullet (s \bullet \varphi) = (ts) \bullet \varphi = s \bullet (t \bullet \varphi);$$
2. Distributivity with respect to superposition:
$$t \bullet (\varphi_1 + \varphi_2) = t \bullet \varphi_1 + t \bullet \varphi_2;$$
3. Weak continuity:
$$t \bullet \varphi \Rightarrow \varphi \qquad t \uparrow 1.$$

General Markov branching process

Definition

A time homogeneous Markov process Ψ_t^φ on $\mathcal{N}$ with $\varphi \in \mathcal{N}$ being a starting configuration, is a Markov branching process if its transition kernel $\{P_t(\varphi, \cdot)\}$ satisfies the Independent branching property:

$$P_t(\varphi_1 + \varphi_2, \cdot) = P_t(\varphi_1, \cdot) * P_t(\varphi_2, \cdot)$$

for any t and $\varphi_1, \varphi_2 \in \mathcal{N}$.

Structure of branching process

The evolution of a branching process Ψ_t^φ is given by two components:

1. Diffusion: every particle moves independently according to a time homogeneous diffusion process;
2. Branching: after exponential time a particle is replaced independently of other particles by an offspring finite point process Ψ^x (possibly empty) whose distribution may depend on x – the position of the mother particle at the branching time.

Maximality of the branching operation

Theorem (Zanella & Z.'15)

A stochastic operation $\bullet$ on $\mathcal{N}$ is associative and distributive if and only if it is a branching operation, i.e.

$$e^{-t} \bullet \varphi \overset{D}{=} \Psi_t^\varphi.$$

for some general Markov branching process Ψ_s^φ.

NB

Characterise stable and SD PPs with respect to general branching operation. Conditions on local finiteness of Φ in the case $\bullet \neq \circ$ are also required.

References

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