Curvature measures of random sets - A survey

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0. Introduction

Background

Developments in classical and modern geometry concerning complete systems of Euclidean invariants with certain geometric properties

Applications to random models in former and recent literature

Content

1. Curvature measures
2. Stationary point processes of geometric sets
3. Stationary random cell complexes and tessellations
4. Fractal models

1. Curvature measures

Differential geometry:
integrals of k th mean curvatures of a d-dimensional submanifold $M_d \subset \mathbb{R}^d$ with smooth boundary:

$$C_k(M_d) := \int_{\partial M_d} S_{d-1-k}(\varkappa_1, \ldots, \varkappa_{d-1}) \, d\mathcal{H}^{d-1}$$

k th Lipschitz-Killing curvature, $k = 0, \ldots, d-1$, where

$$S_l((\varkappa_1, \ldots, \varkappa_{d-1}) := \mathrm{const}(d,l) \sum_{1 < i_1 \ldots \leq i_l < d-1} \varkappa_1 \ldots \varkappa_l$$

l th symmetric function of principal curvatures $\varkappa_1, \ldots, \varkappa_{d-1}$

Special cases: $k = 0$ total Gauss curvature = Euler characteristic, $k = d-2$ total mean curvature, $k = d-1$ surface area, define additionally for $k = d$: $C_d(M_d) := \mathcal{L}^d(M_d)$ volume

Measure versions for j-dimensional submanifolds: $C_k(M_j, \cdot)$, $1 \leq j \leq d$
Extensions to piecewise flat spaces via Morse index theory

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Convex geometry:

$V^k(K)$ k th intrinsic volume of a convex body K; for smooth boundary

$$V^k(K) = C_k(\partial K)$$

Additive extensions to the convex ring $\mathcal{C}$ (finite unions of compact convex sets in $\mathbb{R}^d$):

[Hadwiger 1957], [Groemer 1978], [Schneider 1980] (measure version), [McMullen/Schneider 1983], [Klain 1995]: ideas from convex integral geometry, C_k as motion invariant valuations on $\mathcal{C}$, "Inclusion-exclusion-principle"

Geometric measure theory (extension of both approaches):

integrals of k th generalized mean curvatures over the unit normal bundle norM_d of a d-dimensional submanifold $M_d \subset \mathbb{R}^d$ with positive reach (unique foot point property)

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l-th symmetric function of generalized principal curvatures $\varkappa_1, \ldots, \varkappa_{d-1}$
Curvature measures: [Federer 1959], above explicit representation [Z. 1986]

Additive extension to unions of sets with positive reach: [Z.1987], [Rataj, Z. 2001]; to other classes of sets: subanalytic sets [Fu 1994], o-minimal sets [Bröcker/Kuppe 2000], [Bernig 2005] (via stratified Morse theory), Lipschitz domains of bounded curvature [Rataj/Z.2005],..., WDC-sets (Pokorny/Rataj/Zajicek 2017)

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2. Stationary point processes of geometric sets (germ-grain processes)

Convex sets

[Kendall 1974], [Matheron 1975], [Davy 1976], [Ripley 1976], [Stoyan 1979], [Kellerer 1884], [Weil 1983, 1984], [Weil/Wieacker 1984], [Stoyan/Kendall/Mecke 1987], [Mecke/Schneider/Stoyan/Weil 1990], [Schneider/Weil 2000], ...

Manifolds

[Fava/Santalo 1979]

Sets with positive reach

Z. (1986), ...

Many applications of integral-geometric relationships in stereology

Examples for the case of motion invariant germ-grain processes:
1. Relationships between the curvature densities c_k of the random j-th curvature measures C_j of the union of all grains and those of the intersection with a fixed p-dimensional plane E_p, say $c_k^{E_p}$,

$$c_k^{E_p} = \gamma(d+k-p, p, d)\, c_{d+k-p}$$

2. For the special case of Poisson processes:

$$c_k = -\exp(-\lambda\overline{\mathbf{C}}_d)\sum_{s=1}^{d-k}\frac{(-1)^s}{s!}\sum_{\substack{r_1+\cdots r_s\\=(s-1)d+k}}\prod_{j=1}^{s}\left(\Gamma\left(\frac{r_j+1}{2}\right)\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{-1}\lambda\overline{\mathbf{C}}_{r_j}\right),$$

$$\text{if}\quad k \le d-1\,,\quad \text{and}\quad c_d = 1-\exp(-\lambda\overline{\mathbf{C}}_d)\,,$$

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3. Stationary random cell complexes and tessellations

Random tessellations generated by hyperplanes, random mosaics with convex cells:
First papers: [Ambartzumian 1970,1974], [Miles 1976], [Cowan 1978,1980], [Mecke 1980,1984], [Santalo 1984], ... , large literature up to now.

Stationary random mosaics [Weiss/Z. 1988] and more general cell complexes [Z. 1988] with non-smooth cells:

Mean value relationships [Z. 1988]:

$$c_k^i = \sum_{j=k}^{i} (-1)^{j-k} N^j \overline{\mathbf{C}}_k^j \,, \quad \overline{\mathbf{C}}_k^i = (-1)^{i-k} (N^i)^{-1} (c_k^i - c_k^{i-1}) \,,$$

where c_k^i is the k-th curvature density of the random i-skeleton, N^i the mean number of i-cells per unit volume, and $\overline{\mathbf{C}}_k^i$ is the mean k-th curvature of the typical i-cell, extensions to curvature-direction measures

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4. Fractal models

For $\varepsilon > 0$ and $A \subset \mathbb{R}^d$ denote

$$A_\varepsilon := \{x \in \mathbb{R}^d : d(x, A) \leq \varepsilon\}.$$

Theorem [Fu 1985]

For any compact $K \subset \mathbb{R}^d$ with $d \leq 3$, Lebesgue-a.e. $\varepsilon > 0$ is a regular value of the distance function of K and, hence, the closure of the complement of the the parallel set K_ε has positive reach.

For arbitrary d and compact K with this property define the k th Lipschitz-Killing curvature of the parallel sets K_ε for those ε by

$$C_k(K_\varepsilon) := (-1)^{d-k} C_k\left(\overline{(K_\varepsilon)^c}\right)$$

(consistent definition).

For classical sets K as above we have

$$\lim_{\varepsilon \to 0} C_k(K_\varepsilon) = C_k(K),$$

for fractal sets explosion. Therefore rescaling is necessary:

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Fractal curvatures in the deterministic case [Winter 2008]: The limits

$$C_k^{frac}(F) := \lim_{\varepsilon \to 0} \varepsilon^{D-k} C_k(F_\varepsilon)$$

or, more generally,

$$C_k^{frac}(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-k} C_k(F_\varepsilon) \frac{1}{\varepsilon} \, d\varepsilon$$

exist for certain classes of self-similar fractal sets F of Hausdorff dimension D. (Integral representation for $C_k(F)$ which admits some explicit or numerical calculations.)

Assumptions: open set condition, polyconvex parallel sets.

New system of geometric parameters, allows to distinguish self-similar fractals with equal Hausdorff dimension, but different geometric features.

Extensions for non-polyconvex parallel sets are included in the stochastic version below.

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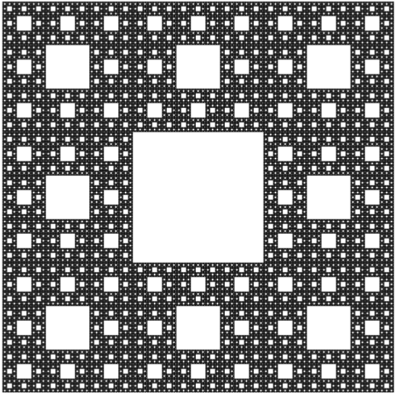
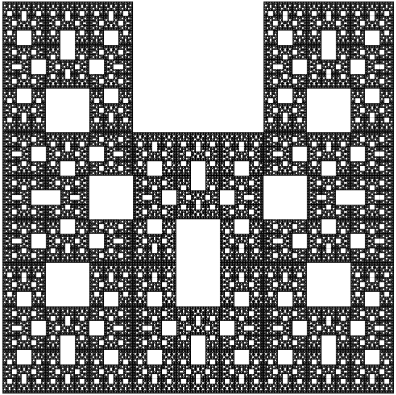
two self-similar sets with the same Hausdorff dimension $\ln 8/\ln 3$



Sierpinski carpet

$C_0^{frac} = -0,016,\ C_1^{frac} = 0,0725$



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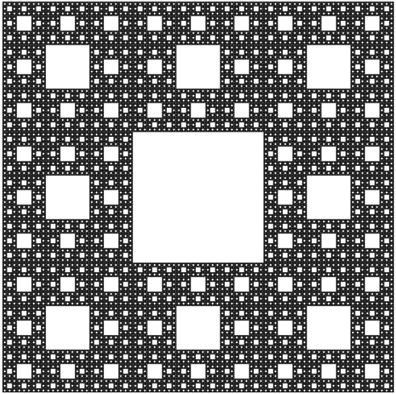

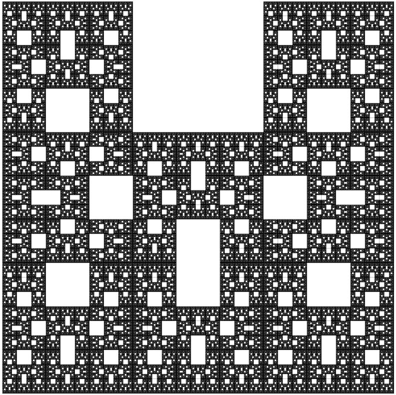

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Self-similar random sets:

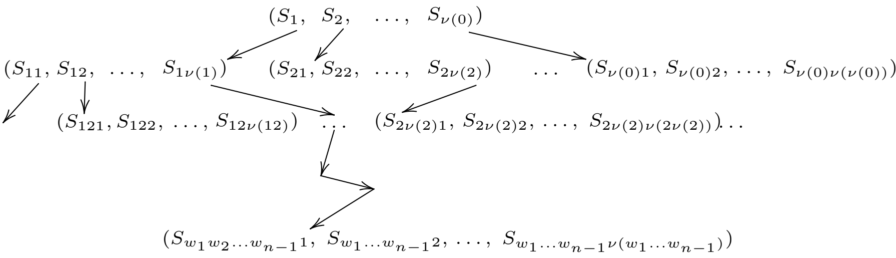
J fixed compact set in $\mathbb{R}^d$ with $\overline{\mathrm{int}(J)} = J$

$(S_1, \ldots, S_\nu)$ random number of random contracting similarities such that

1. $S_0 := \mathrm{id}$ for $\nu = 0$
2. $1 < \mathbb{E}\nu < \infty$ (supercritical case)
3. $S_i(J) \subset J$ and $S_i(\mathrm{int}(J)) \cap S_j(\mathrm{int}(J)) = \emptyset$, $i \neq j$, w.p.1

   (open set condition OSC)

Galton-Watson tree of random similarities:



i.i.d. copies of $(S_1, \ldots, S_\nu)$

Construction of the self-similar random set F:

$$F = \bigcap_{n=1}^{\infty} \bigcup_{w = w_1 \ldots w_n \in W_n} S_{w_1} \circ S_{w_1 w_2} \circ \ldots \circ S_{w_1 \ldots w_n}(J)$$

where J basic compact set (from OSC) and inductively,
$W_0 := \emptyset$,
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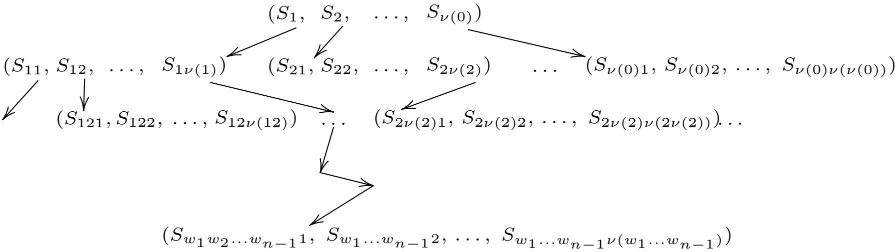

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F is stochastically self-similar:

$$F \overset{(d)}{=} \bigcup_{i=1}^{\nu} S_i(F^i)$$

where $F^1, F^2, \ldots$ are i.i.d. copies of F, independent of the independent pair $(F, (S_1, \ldots, S_\nu))$

Hausdorff dimension D of the random fractal set F a.s. determined by:

$$\mathbb{E}\left(\sum_{i=1}^{\nu} {r_i}^D\right) = 1$$

where r_i random contraction ratio of the random similarity S_i

(Falconer, Graf, Mauldin/Williams [1986,87] under SOSC and Patzschke [1997] general case)

Gatzouras [2000] Minkowski content of F: a.s. (average) limit of

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Random fractal curvatures:

Geometric assumptions

(I) reach $\overline{(F_r)^c} > 0$ for Lebesgue-a.a. $r > 0$ w.p.1

(fulfilled in $\mathbb{R}^d$ with $d \leq 3$, for $d > 3$ and polyconvex parallel sets, conjecture: always)

for such r the random Lipschitz-Killing curvatures $C_k(F_r),\ k = 0, \ldots, d$, and their local variants, random measures $C_k(F_r, \cdot)$, are determined

(II) $\mathbb{E}\left(\sup_{r\geq 1}\left\{r^{-k}|C_k(F_r)|\right\}\right) < \infty$

(III) $\mathbb{E}\left(\sup_{w,w',0<\varepsilon<1}\left\{\varepsilon^{-k}C_k^{var}(F_\varepsilon, (\bar{S}_w(J))_\varepsilon \cap \bar{S}_{w'}(J))_\varepsilon\right\}\right) < \infty$
where $\bar{S}_w(J)$ and $\bar{S}_{w'}(J)$ copies of J of size nearly ε under the above tree of similarities (regular overlapping)

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(III) $\mathbb{E}\left(\sup_{w,w',0<\varepsilon<1}\left\{\varepsilon^{-k} C_k^{var}(F_\varepsilon, (\bar{S}_w(J))_\varepsilon \cap \bar{S}_{w'}(J))_\varepsilon\right\}\right) < \infty$
where $\bar{S}_w(J)$ and $\bar{S}_{w'}(J)$ copies of J of size nearly ε under the above tree of similarities (regular overlapping)

((II) - (III) are fulfilled for polyconvex parallel sets, but also for other classes, e.g. Koch curve or sponge-type fractals)

Random fractal curvatures:

Geometric assumptions

(I) reach $\overline{(F_r)^c} > 0$ for Lebesgue-a.a. $r > 0$ w.p.1

(fulfilled in $\mathbb{R}^d$ with $d \leq 3$, for $d > 3$ and polyconvex parallel sets, conjecture: always)

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Known before: The following limit exists a.s. (martingale convergence theorem).

$$X := \lim_{n\to\infty} \sum_{w\in W_n} r_{w_1}^D r_{w_1 w_2}^D \dots r_{w_1 w_2 \dots w_n}^D \quad \text{and} \quad \mathbb{E}X = 1$$

where the sums run over the words of length n (Galton-Watson tree); denote

$$\mu := \mathbb{E}\left(\sum_{i=1}^{\nu} 1_{(\cdot)}(|\ln r_i|)\right) .$$

Main results [Z. 2011] Under the above conditions the following limits exist:

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Special case $k = d$: Minkowski content

Background: classical renewal theorem for expectations, renewal theorem for general random walks in the sense of Jagers (Nerman [1981]) for a.s. convergence

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THANK YOU !