Geometric functionals of fractal percolation

Steffen Winter

Karlsruhe Institute of Technology

joint work with Michael Klatt

19th Workshop on Stochastic Geometry, Stereology and Image Analysis

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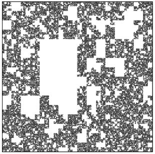

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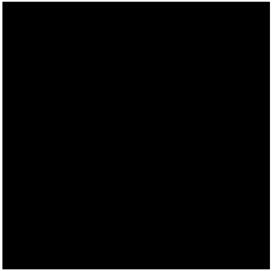

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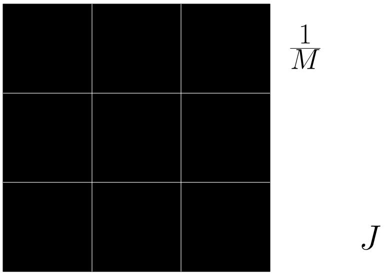

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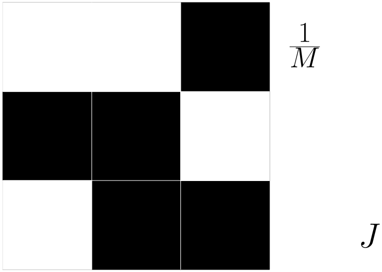

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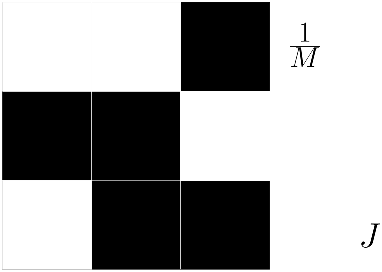

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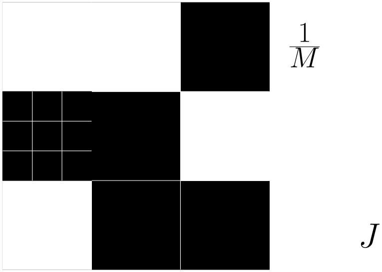

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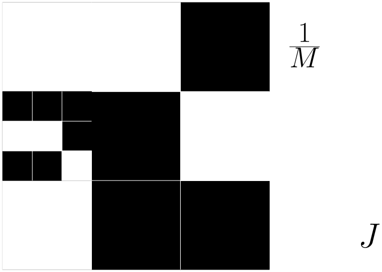

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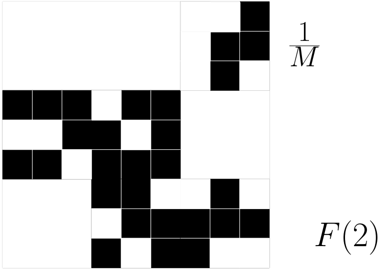

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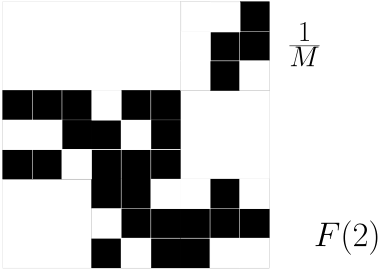

Let $F(n)$ be the union of the squares kept in the n-th step.

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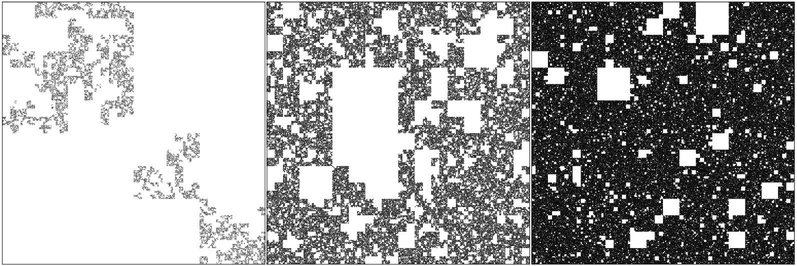

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- further properties and related models: Falconer and Grimmett, Dekking and Meester, Orzechowski, Broman and Camia, ...

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- compare with fractal curvatures [Zähle 11]:

$$\mathcal{C}_k(F) := \lim_{\varepsilon \searrow 0} \varepsilon^{D-k} \mathbb{E} V_k(F_\varepsilon).$$

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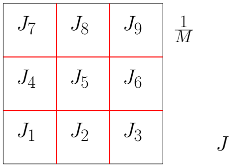
$$V_k([0,1]^d) + \sum_{T\subset\{1,\ldots,M^d\}, |T|\geq 2} (-1)^{|T|-1} \sum_{n=1}^{\infty} r^{n(D-k)} \mathbb{E} V_k\Big(\bigcap_{j\in T} F^j(n)\Big).$$

$F^j(n)$ is the union of the level-n cubes contained in J_j.



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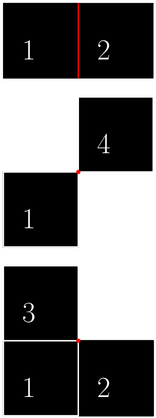

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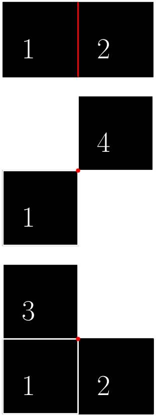

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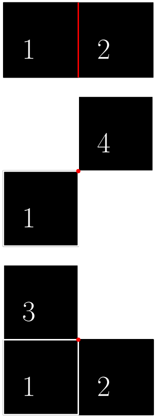

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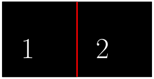

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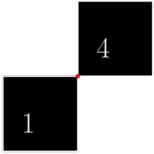

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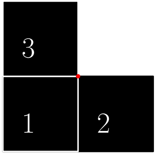

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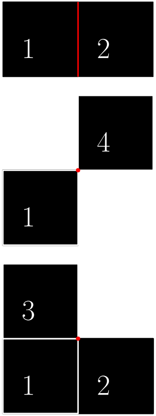

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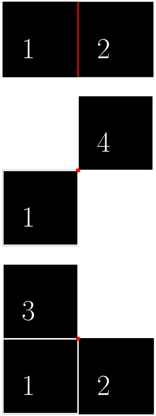

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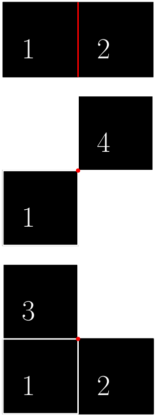

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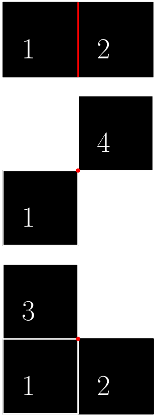
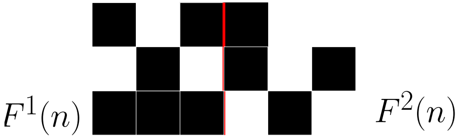
$$E_4 := (M-1)^2\sum_{n=1}^{\infty} r^{nD}p^{4n}$$



Reducing the dimension

- Let K_1, K_2 be two independent (1-dim.) fractal percolations defined on $I = [0, 1]$ (with the same M and p as F).



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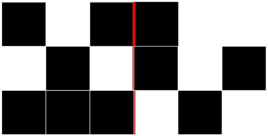
$F^1(n)$  $F^2(n)$

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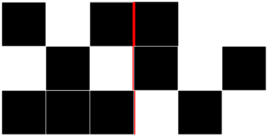
$F^1(n)$  $F^2(n)$

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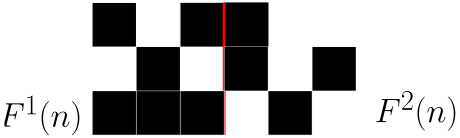
Then for each $n \in \mathbb{N}$, in distribution

$$F^1(n) \cap F^2(n) = \psi(\tilde{K}_1(n-1) \cap \tilde{K}_2(n-1)),$$

where $\psi : I \to \mathbb{R}^2$ is the similarity mapping I to $J_1 \cap J_2$.



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In particular,

$$\mathbb{E}V_k(F^1(n) \cap F^2(n)) = r^k p^2 \mathbb{E}V_k(K_1(n-1) \cap K_2(n-1)).$$

Expected Euler characteristic ($d = 2$)

$$\mathcal{Z}_0(F) = 1 - E_1 + (M-1)^2 \left(-\frac{2p}{M^2 - p} + \frac{4p^2}{M^2 - p^2} - \frac{p^3}{M^2 - p^3} \right),$$

$$E_1 = 2(M-1)^2 \left(\left(\frac{3}{M-1} - \frac{4p}{M-p} + \frac{p^2}{M-p^2} \right) \frac{p}{M-p} - \frac{2Mp}{(M-1)(M^2-p)} + \frac{4Mp^2}{(M-p)(M^2-p^2)} - \frac{Mp^3}{(M-p^2)(M^2-p^3)} \right)$$

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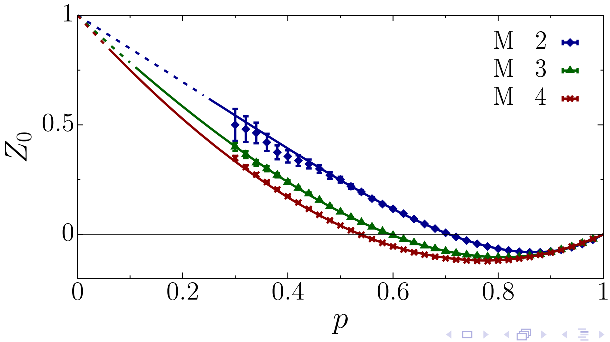

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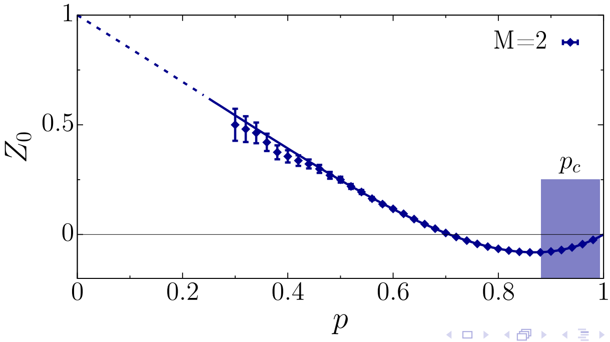

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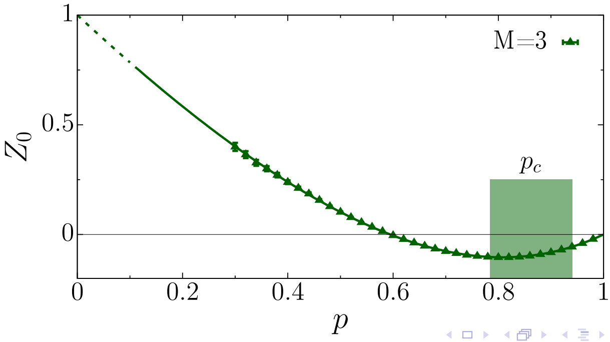
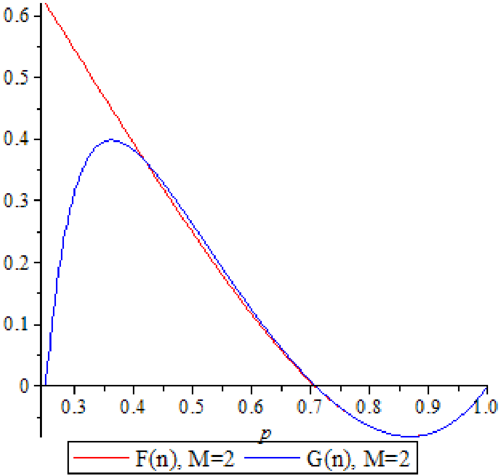


Another approximation of F

Instead of the sequence $F(n)$ consider the union $G(n)$ of those boxes Q of level n, which have a nonempty intersection with F. Then

- $G(n) \subseteq F(n)$,
- $F = \bigcap_n G(n)$;
- $G(n)$ are still polyconvex;
- similar formulas hold for the rescaled limits of expected intrinsic volumes;
- the survival probability $\tilde{p} = \tilde{p}(p) := \mathbb{P}(F \neq \emptyset)$ appears

Approximation by $F(n)$ vs. $G(n)$



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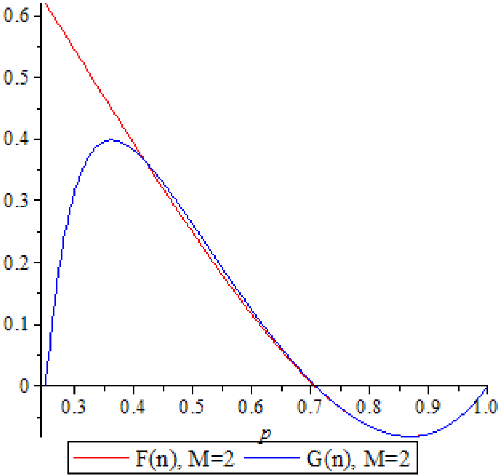

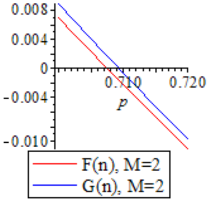

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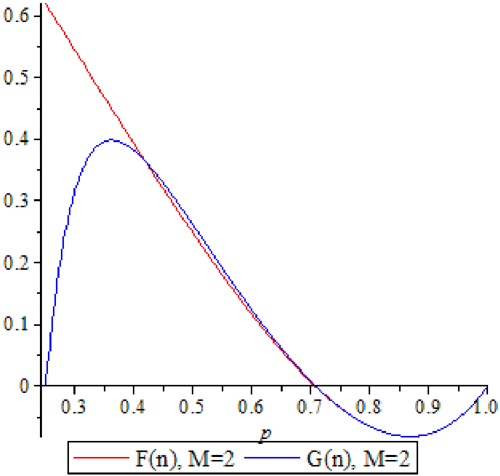
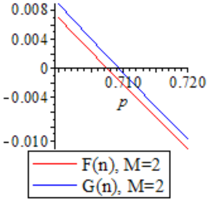
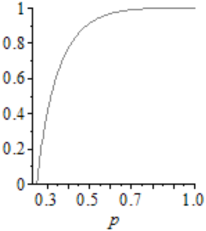
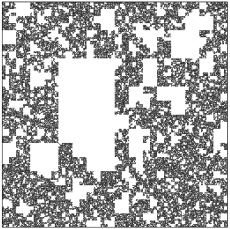


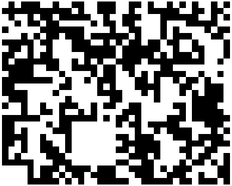


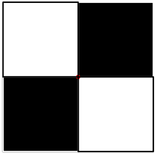


Yet another approximation of F

In $F(n)$ diagonal connections are counted, which produce many extra holes.







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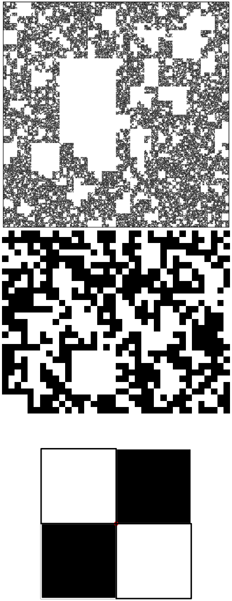

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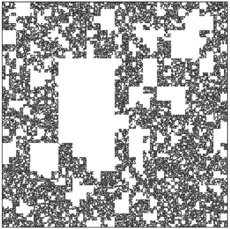
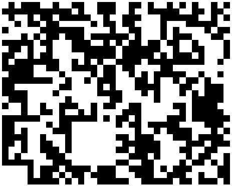
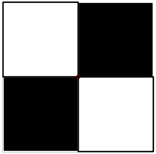
In $F(n)$ diagonal connections are counted, which produce many extra holes. But diagonal connections do not survive in F.

Consider the closed complements of $F(n)$ in $J = [0,1]^d$:

$$C(n) = \overline{J \setminus F(n)}.$$







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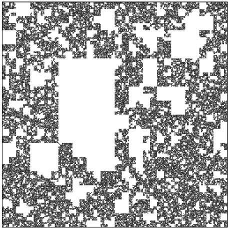

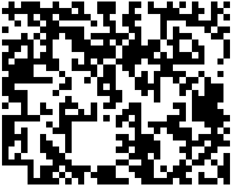

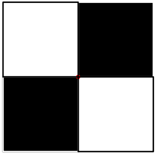

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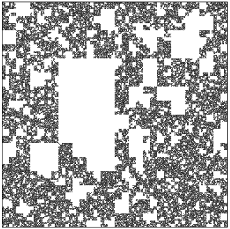
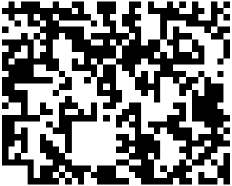
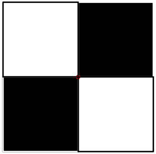
$$\mathcal{Y}_k(F) := \lim_{n\to\infty} r^{n(D-k)} \mathbb{E} V_k(C(n))$$







Open Questions

- Is the zero of the rescaled limit of expected Euler characteristics a lower bound for p_c?
  Does $\mathcal{Z}_0(F_p) > 0$ imply $p < p_c$?

$$\mathcal{Z}_0(F_p) := \lim_{n\to\infty} r^{Dn} V_0(F_p(n))$$

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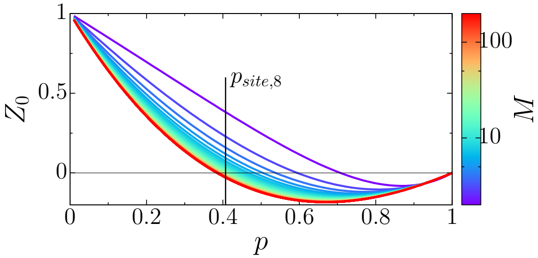

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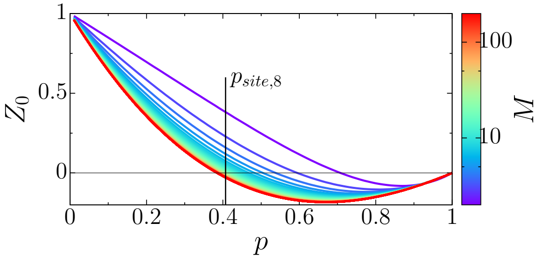

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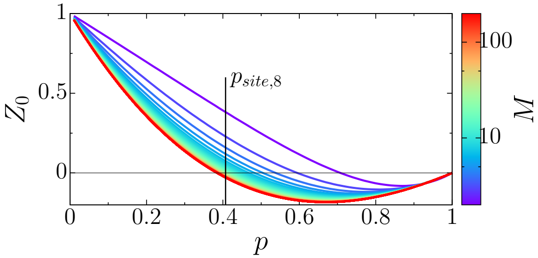
- similar relations for other scale invariant models (Boolean multiscale models, Brownian loop soup, ...) [Broman,Camia 10]



Deadline for contributions: 31 May 2017