Convex Hulls of Lévy Processes

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Basics

- For a non-empty closed convex set $K \subset \mathbb{R}^d$, the support function is

$$h(K,u) = \sup\{\langle x,u\rangle : x \in K\}, \quad u \in \mathbb{R}^d.$$

- For two sets K and L in $\mathbb{R}^d$, the Minkowski sum is denoted by $K + L = \{x + y : x \in K, y \in L\}$.

- It is known that the volume $V_d(K + tL)$ is a polynomial in $t \geq 0$ of degree d. The mixed volumes $V(K[j], L[d-j])$, $j = 0, \dots, d$, appear as coefficients of this polynomial, so that

$$V_d(K + tL) = \sum_{j=0}^{d} \binom{d}{j} t^{d-j} V(K[j], L[d-j]), \quad t \geq 0.$$

  Note that the mixed volume is a function of d arguments, and $K[j]$ stands for its j arguments, all being K.

- The intrinsic volumes of a convex body K are normalised mixed volumes

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B^d[d-j]), \quad j = 0, \dots, d,$$

  where B^d denotes the centred d-dimensional unit ball and κ_{d-j} is the volume of B^{d-j}.

Setting

- Let $X(t)$, $t \geq 0$, be a *Lévy process* in $\mathbb{R}^d$ starting at the origin.
- Z_s denotes the closed convex hull of $\{X(t) : 0 \leq t \leq s\}$.
- The *Lévy measure* of the process $X(t)$, $t \geq 0$, is denoted by ν.

Known results

- Kampf et. al. (2012) considered symmetric α-stable Lévy processes in $\mathbb{R}^d$ with $\alpha > 1$.
  - A formula for $\mathbf{E} V_1(Z_s)$ was obtained.
  - If the process has independent coordinates, then all intrinsic volumes of Z_s are integrable.
- Eldan (2014) considered the Brownian Motion in $\mathbb{R}^d$. He obtained an explicit formula for the expected value of all intrinsic volumes of Z_s.

Integrability

Define

$$\beta_\nu = \sup \left\{ \beta > 0 : \int_{\|x\|>1} \|x\|^\beta \nu(dx) < \infty \right\}.$$

Theorem

If $0 \leq p < \beta_\nu$, then $\mathbf{E} V_j(Z_s)^p < \infty$ for all $j = 0, \ldots, d$ and all $s \geq 0$.

Corollary

If X is the Brownian motion, then $\mathbf{E} V_j(Z_s)^p < \infty$ for all $p \geq 0$, all $j = 0, \ldots, d$ and all $s \geq 0$.

Idea of the proof

- We prove that $\mathbf{E} V_d(Z_s + B^d)^p < \infty$, since $V_d(Z_s + B^d) = \sum_{j=0}^{d} \kappa_{d-j} V_j(Z_s)$.
- The main idea is to split the path of the Lévy process into several parts with integrable volumes of their convex hulls.
- We cover the path with unit balls.

Idea of the proof II

- We define the renewal process N_s which counts how many unit balls are needed to cover the whole path up to s.
- Let $I_1, \dots, I_{N_s}$ denote the translated (to the origin) line segments between two centres of the unit balls which are near to each other.
- We obtain that $Z_s \subseteq I_1 + \cdots + I_{N_s} + B^d$.
- Therefore, we prove that $V_d(I_1 + \cdots + I_{N_s} + 2B^d)^p$ is integrable.

Random parallelepiped

- The j-dimensional volume of the parallelepiped spanned by $u_1, \dots, u_j \in \mathbb{R}^d$ is denoted by $D_j(u_1, \dots, u_j)$.
- Let Y be a random compact set in $\mathbb{R}^d$. Its selection expectation $\mathbf{E}Y$ is defined as the convex body with support function $\mathbf{E}h(Y, u)$, $u \in \mathbb{R}^d$.

Theorem

Let $j \in \{1, \dots, d\}$. If $\xi_1, \dots, \xi_j \in \mathbb{R}^d$ are independent integrable random vectors, then

$$V(\mathbf{E}[0, \xi_1], \dots, \mathbf{E}[0, \xi_j], B^d[d-j]) = \frac{(d-j)!}{d!} \kappa_{d-j} \mathbf{E} D_j(\xi_1, \dots, \xi_j).$$

Random determinants

- Let M_j be a $d \times j$ matrix composed of j columns being i.i.d. copies of a random vector $\xi \in \mathbb{R}^d$.

Corollary

If $\xi \in \mathbb{R}^d$ is an integrable random vector, then

$$\mathbf{E}\sqrt{\det M_j^T M_j} = j!\,V_j(\mathbf{E}[0,\xi]), \quad j = 1, \ldots, d.$$

Stable processes

- The characteristic function of a symmetric α-stable Lévy process $X(t)$, $t \geq 0$, with $\alpha > 1$ can be represented as

$$\mathbf{E} \exp\{\imath \langle X(t), u\rangle\} = \exp\left\{-t h(K,u)^{\alpha}\right\}, \quad u \in \mathbb{R}^d, t \geq 0,$$

where

$$h(K,u) = \sup\{\langle x, u\rangle : x \in K\}, \qquad u \in \mathbb{R}^d,$$

is the support function of a convex body K called the associated zonoid of $X(1)$.

Expected intrinsic volumes

Theorem

Let $X(t)$, $t \geq 0$, be a symmetric α-stable Lévy process in $\mathbb{R}^d$ with $\alpha > 1$. Then, for all $j = 1, \ldots, d$,

$$\mathbf{E} V_j(Z_s) = \frac{\Gamma(1-1/\alpha)^j \Gamma(1/\alpha)^j}{\pi^j \Gamma(j/\alpha + 1)} V_j(K) s^{j/\alpha},$$

where K is the associated zonoid of $X(1)$.

Idea of the proof

- Since Z_s is self-similar, it is enough to prove the Theorem for $s = 1$.
- The main idea is to approximate the Lévy process with the random walk $S_i = X(i/n)$, $i = 0, \dots, n$.
- We use a known formula for the expected intrinsic volumes of the convex hull of a random walk, see Vysotsky and Zaporozhets (2015).
- The auxiliary result on random determinants is used.

Expected intrinsic volumes II

Example

If X is the standard Brownian motion, then $\alpha = 2$ and $K = \frac{1}{\sqrt{2}} B^d$, so we recover

$$\mathbf{E} V_j(Z_s) = \binom{d}{j} \left(\frac{\pi}{2}\right)^{j/2} \frac{\Gamma((d-j)/2+1)}{\Gamma(j/2+1)\Gamma(d/2+1)} s^{j/\alpha}, \quad j = 1, \ldots, d,$$

see Eldan (2014).

Example

If $X(1)$ is spherically symmetric, then $K = c^{1/\alpha} B^d$, $c > 0$, so that

$$\mathbf{E} V_j(Z_s) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} \frac{\Gamma(1-1/\alpha)^j \Gamma(1/\alpha)^j}{\pi^j \Gamma(j/\alpha+1)} (cs)^{j/\alpha}, \quad j = 1, \ldots, d.$$

Interior of the convex hull

Theorem

Let $X(t)$, $t \geq 0$, be a symmetric Lévy process in $\mathbb{R}^d$, such that $\langle X(1), u \rangle$ has a non-atomic distribution for each $u \neq 0$. Then, $\mathbf{P}\{0 \in \operatorname{Int} Z_s\} = 1$ and $\mathbf{P}\{X(s) \in \operatorname{Int} Z_s\} = 1$ for each $s > 0$.

Limit theorem

- We denote by T_1 the first exit time of the process X from the unit ball.

Theorem

Assume that $X(t)$, $t \geq 0$, is a Lévy process in $\mathbb{R}^d$ such that $X(T_1)$ lies in the domain of attraction of a strictly α-stable random vector η, that is the sum of n i.i.d. copies of $X(T_1)$ scaled by $(n^{1/\alpha}\ell(n))^{-1}$ with a slowly varying function ℓ converges in distribution to η. Then $(t^{1/\alpha}\ell(t))^{-1}Z_t$ converges in distribution to $\mathrm{conv}\{Y(s) : 0 \leq s \leq (\mathbf{E}T_1)^{-1}\}$, *where Y is a Lévy process such that $Y(1)$ coincides in distribution with η.*

References

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