Integral Geometric Formulae for Tensorial Curvature Measures

(based on joint work with Daniel Hug)

Jan A. Weis

Karlsruhe Institute of Technology

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Outline

Tensorial curvature measures

Integral geometric formulae

Kinematic and Crofton formulae for tensorial curvature measures

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Intrinsic volumes

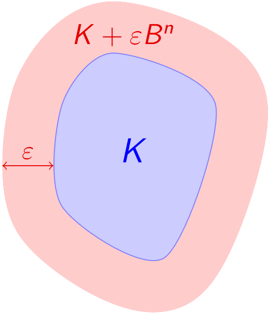
The intrinsic volumes

$$V_j : \mathcal{K}^n \to \mathbb{R}, \qquad j \in \{0, \dots, n\},$$

are defined as the coefficients of the monomials in the Steiner formula

$$\mathcal{H}^n(K + \varepsilon B^n) = \sum_{j=0}^{n} \kappa_{n-j} \, V_j(K) \, \varepsilon^{n-j},$$

for a convex body $K \in \mathcal{K}^n$ and $\varepsilon \geq 0$.



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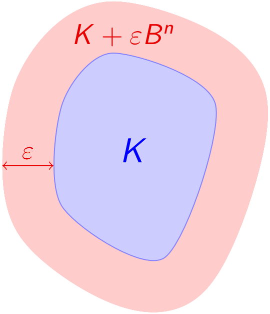

They are classical examples of valuations in convex geometry, i.e.

$$V_j(K) + V_j(K') = V_j(K \cup K') + V_j(K \cap K')$$

whenever $K, K', K \cup K' \in \mathcal{K}^n$.

Support and curvature measures

The support measures

$$\Lambda_j : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}) \to \mathbb{R}, \qquad j \in \{0, \dots, n-1\},$$

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- weakly continuous: If $K_i \to K$ (in the Hausdorff metric) then

$$\Lambda_j(K_i, \cdot) \xrightarrow{w} \Lambda_j(K, \cdot).$$

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The curvature measures ϕ_j, $j \in \{0, \dots, n-1\}$, are the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of the support measures, and hence defined by

$$\phi_j(K, \cdot) := \Lambda_j(K, \cdot \times \mathbb{S}^{n-1}).$$

Curvature measures on polytopes

The curvature measures of a polytope $P \in \mathcal{P}^n$ are explicitly given by

$$\phi_j(P,\beta) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \int_{F \cap \beta} \mathcal{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} \mathcal{H}^{n-j-1}(\mathrm{d}u)$$

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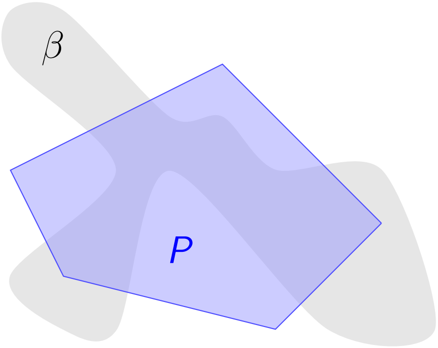

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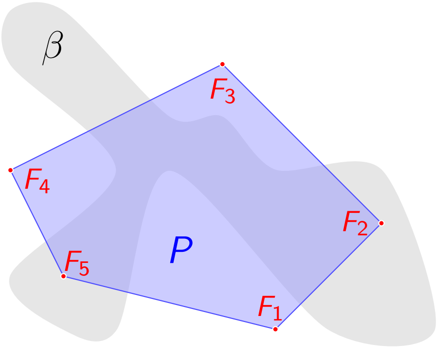

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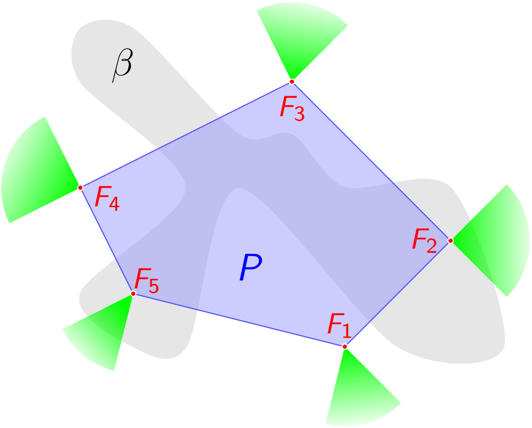

(Generalized) tensorial curvature measures

The tensorial curvature measures are a tensor-valued generalization of the curvature measures

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Here $c_{n,j}^{r,s,l} > 0$ and $x^r u^s \in \mathbb{T}^{r+s}$ is a symmetric tensor product, i.e. a symmetric $r + s$-linear mapping from $(\mathbb{R}^n)^{r+s}$ to $\mathbb{R}$.

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On polytopes, we further obtain the generalized tensorial curvature measures from the curvature measures

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On polytopes, we further obtain the generalized tensorial curvature measures from the curvature measures, i.e. they are given by

$$\phi_j^{r,s,l}(P,\beta) := \frac{c_{n,j}^{r,s,l}}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} Q(F)^l \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u),$$

for $r, s, l \in \mathbb{N}_0$, where $Q(F) \in \mathbb{T}^2$ denotes the metric tensor on F.

Properties of the tensorial curvature measures

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- To date, there is no characterization result for the tensorial curvature measures using these properties.
- However, the valuations

$$Q^m \phi_j^{r,s,l}, \qquad r + s + 2l + 2m = p$$

are essentially linearly independent.

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Tensorial curvature measures

Integral geometric formulae

Kinematic and Crofton formulae for tensorial curvature measures

The principal kinematic formula

Let $K, K' \in \mathcal{K}^n$ and $j \in \{0, \ldots, n\}$. Then

$$\int_{G_n} V_j(K \cap gK')\,\mu(\mathrm{d}g) = \sum_{k=j}^{n} \alpha_{njk}\, V_k(K)\, V_{n-k+j}(K'),$$

where $\alpha_{njk} = \frac{\Gamma(\frac{k+1}{2})\,\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{j+1}{2})\,\Gamma(\frac{n+1}{2})}$.

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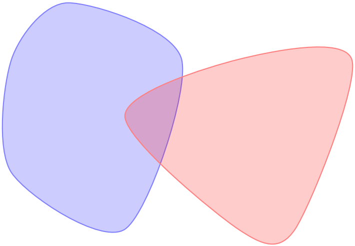

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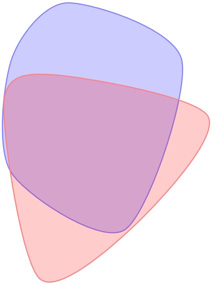

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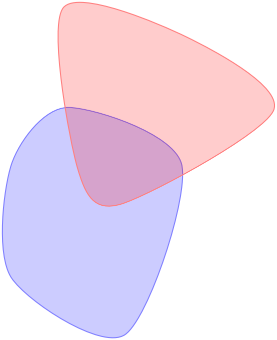

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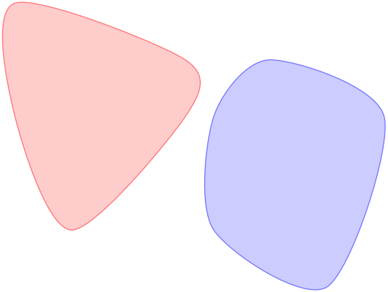

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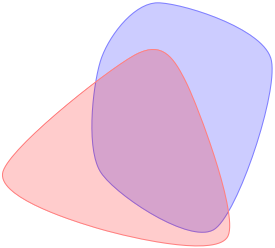

The Crofton formula

Let $K \in \mathcal{K}^n$ and $j, k \in \mathbb{N}_0$ with $j \leq k < n$. Then

$$\int_{\mathsf{A}(n,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) = \alpha_{njk}\, V_{n-k+j}(K),$$

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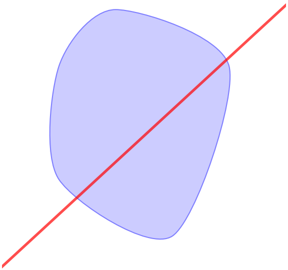

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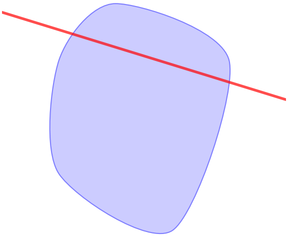

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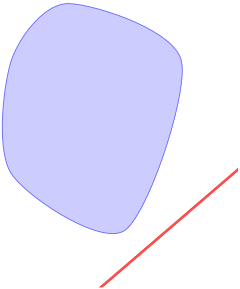

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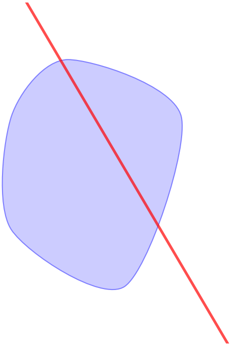

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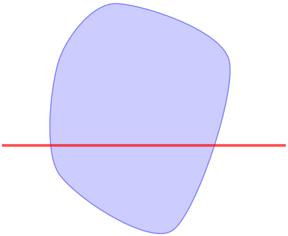

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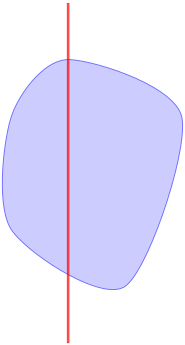

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- In 2016, Svane and Jensen obtained intrinsic Crofton formulae for Minkowski tensors on sets of positive reach (and local versions thereof).

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Outline

Tensorial curvature measures

Integral geometric formulae

Kinematic and Crofton formulae for tensorial curvature measures

Tensorial kinematic formulae

Theorem 1 (Hug and W. '16)

For $P, P' \in \mathcal{P}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$, $j, l, r, s \in \mathbb{N}_0$ *with* $j \leq n$, *and* $l = 0$ *if* $j = 0$,

$$\int_{\mathsf{G}_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g)$$

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$$= \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,k}^{s,l,i,m}\, Q^{m-i} \phi_k^{r,s-2m,l+i}(P, \beta) \phi_{n-k+j}(P', \beta'),$$

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where

$$c_{n,j,k}^{s,l,i,m} := \frac{(-1)^i}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})} \alpha_{njk}.$$

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Tensorial kinematic formulae

Corollary 2 (Hug and W. '16)

For $K, K' \in \mathcal{K}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ *and* $j, r, s \in \mathbb{N}_0$ *with* $j \leq n$,

$$\int_{G_n} \phi_j^{r,s,0}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{1} c_{n,j,k}^{s,0,i,m}\, Q^{m-i} \phi_k^{r,s-2m,i}(K, \beta) \phi_{n-k+j}(K', \beta').$$

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Remarkably, the coefficients $c_{n,j,k}^{s,0,i,m}$ in Theorem 1 vanish, for $i > 1$, due to the quotient

$$\tfrac{(i-2)!}{(-2)!} = \tfrac{\Gamma(i-1)}{\Gamma(-1)} = (-1)^i \tfrac{1}{\Gamma(2-i)} = \mathbb{1}\{i = 0\} - \mathbb{1}\{i = 1\}.$$

Hence, only the generalized tensorial curvature measures with continuous extensions remain in the representation in Corollary 2.

Tensorial kinematic formulae – Sketch of proof

We decompose the motion $g \in G_n$ into a rotation $\vartheta \in \mathrm{SO}(n)$ and a translation by $t \in \mathbb{R}^n$ to get

$$\int_{\mathsf{G}_n} \phi_j^{r,s,l}\Big(P \cap gP', \beta \cap g\beta'\Big)\, \mu(\mathrm{d}g)$$
$$= \frac{c_{n,j}^{r,s,l}}{\omega_{n-j}} \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \phi_j^{r,s,l}\Big(P \cap (\vartheta P' + t), \beta \cap (\vartheta\beta' + t)\Big)\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta)$$

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We decompose the motion $g \in G_n$ into a rotation $\vartheta \in \mathrm{SO}(n)$ and a translation by $t \in \mathbb{R}^n$ and apply the definition of $\phi_j^{r,s,l}$ to get

$$\int_{\mathrm{G}_n} \phi_j^{r,s,l}\Big(P \cap gP', \beta \cap g\beta'\Big)\,\mu(\mathrm{d}g)$$
$$= \tfrac{c_{n,j}^{r,s,l}}{\omega_{n-j}} \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \sum_{G \in \mathcal{F}_j(P \cap (\vartheta P' + t))} Q(G)^l \int_{G \cap \beta \cap (\vartheta\beta' + t)} x^r\, \mathcal{H}^j(\mathrm{d}x)$$
$$\times \int_{N(P \cap (\vartheta P' + t), G) \cap \mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u)\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta)$$

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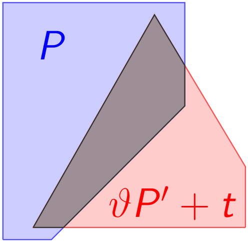
$$\times \int_{N(P \cap (\vartheta P' + t), G) \cap \mathbb{S}^{n-1}} u^s\,\mathcal{H}^{n-j-1}(\mathrm{d}u)\,\mathcal{H}^n(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)$$

For almost all $t \in \mathbb{R}^n$,

$$\mathcal{F}_j(P \cap (\vartheta P' + t)) \ni G = F \cap (\vartheta F' + t)$$

with unique faces $F \in \mathcal{F}_k(P)$ and $F' \in \mathcal{F}_{n-k+j}(P')$.



Tensorial kinematic formulae – Sketch of proof

Therefore,

$$
\begin{aligned}
&\int_{\mathsf{G}_n} \phi_j^{r,s,l}\Big(P \cap gP', \beta \cap g\beta'\Big)\, \mu(\mathrm{d}g) \\
&\quad = \frac{c_{n,j}^{r,s,l}}{\omega_{n-j}} \int_{\mathsf{SO}(n)} \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} Q(F^0 \cap (\vartheta F')^0)^l \\
&\qquad\quad \times \int_{N(P \cap (\vartheta P'+t), F \cap (\vartheta F'+t)) \cap \mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u) \\
&\qquad\quad \times \int_{\mathbb{R}^n} \int_{F \cap (\vartheta F'+t) \cap \beta \cap (\vartheta \beta'+t)} x^r\, \mathcal{H}^j(\mathrm{d}x)\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta).
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Therefore,

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Tensorial kinematic formulae – Sketch of proof

Similar to the proof of the principal kinematic formula, we can simplify the translative part, i.e. the integration with respect to t:

$$I_1 = \frac{c_{n,j}^{r,s,l}}{\omega_{n-j}} \sum_{k=j}^{n} \sum_{F\in\mathcal{F}_k(P)} \int_{F\cap\beta} x^r \, \mathcal{H}^k(\mathrm{d}x) \sum_{F'\in\mathcal{F}_{n-k+j}(P')} \mathcal{H}^{n-k+j}(F'\cap\beta')$$
$$\times \int_{\mathrm{SO}(n)} [F,\vartheta F'] Q \left(F\cap\vartheta F'\right)^l \int_{(N(P,F)+\vartheta N(P',F'))\cap\mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta).$$

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The remaining rotational part, i.e. the integration with respect to ϑ, is the crucial step of the proof.

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The remaining rotational part, i.e. the integration with respect to ϑ, is the crucial step of the proof. It involves

- Grassmannian integration formulae,
- tensor geometry,
- Zeilberger's algorithm.

Tensorial Crofton formulae

Theorem 3 (Hug and W. '16)

Let $P \in \mathcal{P}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $j, k, r, s, l \in \mathbb{N}_0$ with $j < k \leq n$, and with $l = 0$ if $j = 0$. Then,

$$\int_{A(n,k)} \phi_j^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,n-k+j}^{s,l,i,m}\, Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta),$$

where the $c_{n,j,k}^{s,l,i,m}$ are defined as in Theorem 1.

Tensorial Crofton formulae

Corollary 4 (Hug and W. '16)

Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $j, k, r, s \in \mathbb{N}_0$ with $j < k \leq n$. Then,

$$\int_{\mathsf{A}(n,k)} \phi_j^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{1} c_{n,j,n-j+k}^{s,0,i,m}\, Q^{m-i} \phi_{n-k+j}^{r,s-2m,i}(K, \beta),$$

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