Densities of Mixed Volumes for Boolean Models

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(joint work with Daniel Hug (Karlsruhe))

1. Boolean Models

A **Boolean model** (with convex grains) is a random closed set $Z \subset \mathbb{R}^d$, arising as the union of a Poisson particle process X on the space $\mathcal{K}^d$ of convex bodies in $\mathbb{R}^d$,

$$Z = \bigcup_{K \in X} K.$$

If X and Z are **stationary**, X and Z are determined (in distribution) by the **intensity** γ (> 0) and the distribution $\mathbb{Q}$ of the **typical grain**, a probability measure on $\mathcal{K}_0^d \subset \mathcal{K}^d$ of convex bodies with circumcenter at the origin.

A major problem in applications is to estimate γ, the mean number of particles per unit volume, from measurements of the union set Z.

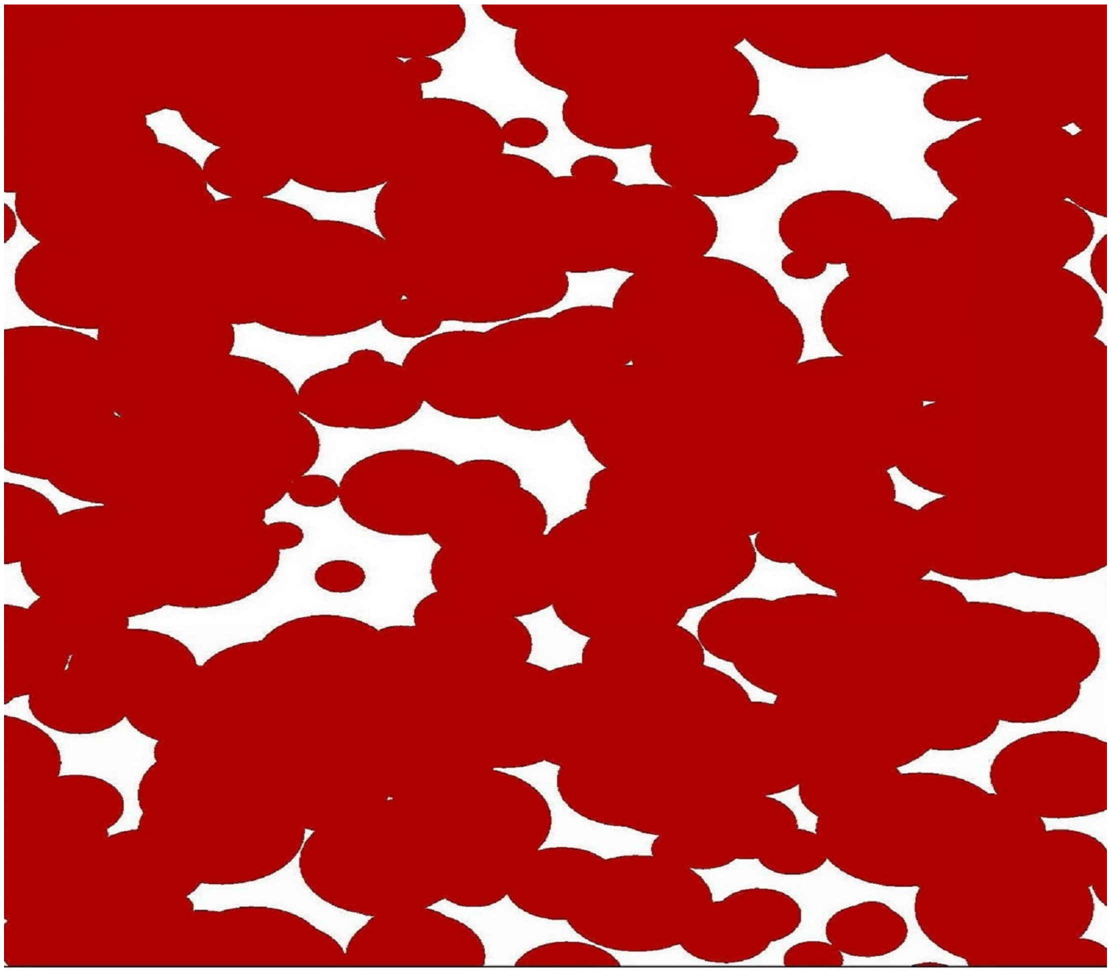
If X and Z are, in addition, **isotropic**, the classical formulas of **Davy and Miles (1978)** allow such an estimation. The formulas express the mean values $\overline{V}_j(Z)$ of the (additively extended) **intrinsic volumes** $V_j, j = 0\ldots, d$, of the Boolean model Z as a triangular array of the mean values $\overline{V}_j(X)$ of X and read

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)},$$

$$\overline{V}_{d-1}(Z) = \mathrm{e}^{-\overline{V}_d(X)}\overline{V}_{d-1}(X),$$

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_d(X)}\Big[\overline{V}_j(X) - \sum_{k=2}^{d-j} \frac{(-1)^k}{k!} \sum_{\substack{m_1,\ldots,m_k=j+1 \\ m_1+\ldots m_k=(k-1)d+j}}^{d-1} c_j^d \prod_{i=1}^{k} c_d^{m_i} \overline{V}_{m_i}(X)\Big],\ j = 0, \ldots, d-2.$$

This system of equations can be inverted from top to bottom to yield $\gamma = \overline{V}_0(X)$ in terms of the mean values $\overline{V}_j(Z)$ for $j = 0, \ldots, d$.



This method does not work for non-isotropic Z anymore, hence mean values for direction dependent functionals have to be considered.

2. Mixed Volumes

Classical directional quantities in Convex Geometry are the **mixed volumes** $V(K[j], M[d-j])$.

For $K, M \in \mathcal{K}^d, \alpha, \beta \geq 0$, we have

$$V_d(\alpha K + \beta M) = \sum_{j=0}^{d} \binom{d}{j} \alpha^j \beta^{d-j} V(K[j], M[d-j]).$$

If $M = B^d$, the unit ball, then $V(K[j], M[d-j]) = c_{jd} V_j(K)$.

Since $V(K[j], M[d-j])$ is continuous and additive in K, it has an extension to polyconvex sets and the limit exists:

$$\overline{V}(Z[j], M[d-j]) = \lim_{r \to \infty} \frac{\mathbb{E} V(Z \cap rB^d[j], M[d-j])}{V_d(rB^d)}.$$

In **W. (2001)** it was shown that these mean values satisfy a variant of the Miles formulas,

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)},$$
$$\overline{V}(Z[d-1], M) = \mathrm{e}^{-\overline{V}_d(X)}\overline{V}(X[d-1], M),$$

and

$$\overline{V}(Z[j], M[d-j]) = \mathrm{e}^{-\overline{V}_d(X)}\Bigg(\overline{V}(X[j], M[d-j])$$
$$- \sum_{k=2}^{d-j} \frac{(-1)^k}{k!} \sum_{\substack{m_1,\ldots,m_k=j+1 \\ m_1+\cdots+m_k=(k-1)d+j}}^{d-1} \overline{V}_{m_1,\ldots,m_k}(X, \ldots, X, M[d-j])\Bigg),$$

for $j = 0, \ldots, d-2$ and $M \in \mathcal{K}^d$.

Here, mean values of mixed expressions occur,

$$\overline{V}_{m_1,\ldots,m_k}(X,\ldots,X,M[d-j])$$
$$= \gamma^k \int \cdots \int V_{m_1,\ldots,m_k}(K_1,\ldots,K_k,M[d-j])\mathbb{Q}(dK_k)\cdots\mathbb{Q}(dK_1)$$

and the mixed functionals $V_{m_1,\ldots,m_k}(K_1,\ldots,K_k,M[d-j])$ arise from the **iterated translation formula** for the mixed volumes,

$$\int_{(\mathbb{R}^d)^{k-1}} V(K_1 \cap (K_2+x_2) \cap \cdots \cap (K_k+x_k)[j], M[d-j])d(x_2,\ldots,x_k)$$
$$= \sum_{\substack{m_1,\ldots,m_k=j+1 \\ m_1+\cdots+m_k=(k-1)d+j}}^{d} V_{m_1,\ldots,m_k}(K_1,\ldots,K_k,M[d-j])$$

and are uniquely determined by the fact that they are homogeneous of degree m_i in K_i, $i=1,\ldots,k$.

Since $V(K[0], M[d]) = V_0(K)V_d(M)$, we obtain, as a special case, the mean value formula for the **Euler characteristic** V_0,

$$\overline{V}_0(Z) = \mathrm{e}^{-\overline{V}_d(X)} \left(\overline{V}_0(X) - \sum_{k=2}^{d} \frac{(-1)^k}{k!} \times \sum_{\substack{m_1,\dots,m_k=1 \\ m_1+\cdots+m_k=(k-1)d}}^{d-1} \overline{V}_{m_1,\dots,m_k}(X,\dots,X) \right). \qquad (1)$$

Thus, in order to determine the intensity $\gamma = \overline{V}_0(X)$ from this equation, the mixed densities $\overline{V}_{m_1,\dots,m_k}(X,\dots,X)$ have to be obtained, for all indices $m_1,\dots,m_k$, by the equations for

$$\overline{V}(Z[j], M[d-j]), \;\; j = 1,\dots,d-1.$$

3. The Main Result

Theorem 1. *Let Z be a stationary Boolean model in $\mathbb{R}^d, d \geq 2$, with convex grains and satisfying the moment condition*

$$\int_{\mathcal{K}_0^d} V_1(K)^{d-2} \, \mathbb{Q}(dK) < \infty.$$

If the densities of the mixed volumes $V(Z[j], M[d-j])$ are given for $j = 0, \ldots, d$ and all $M \in \mathcal{K}^d$, then the intensity γ of the underlying Poisson particle process X is uniquely determined.

For small dimensions $d = 2$ and $d = 3$ this result was shown in a number of papers (**W. (1995, 1999, 2001)**), an approach for $d = 4$ in **W. (2001)** was incomplete, the case $d \geq 5$ remained open. In these papers, it was used that $V(K[1], M[d-1])$ has an integral representation on the unit sphere,

$$V(K[1], M[d-1]) = \frac{1}{d} \int_{S^{d-1}} h^*(K, u)\, S_{d-1}(M, du). \tag{2}$$

Here $h^*(K, \cdot)$ is the centered support function of K and $S_{d-1}(K, \cdot)$ is the $(d-1)$st area measure of K.

Moreover, the value of $V(K[1], M[d-1])$ for fixed K and all $M \in \mathcal{K}^d$ determines $h^*(K, \cdot)$, and for fixed M and all $K \in \mathcal{K}^d$ it determines $S_{d-1}(M, \cdot)$.

As a consequence, all mean values $\overline{V}_{m_1,\ldots,m_k}(X,\ldots,X)$ in (1) are determined by the higher order mean values

$$\overline{V}(Z[j], M[d-j]), \quad j \geq 1, M \in \mathcal{K}^d,$$

as long as the indices m_i are either 1 or $d-1$.

This is sufficient in dimensions $d = 2$ and $d = 3$ (but insufficient in dimension 4, since then $\overline{V}_{2,2}(X, X)$ occurs).

Thus for $d \geq 4$, a decomposition of

$$V(K[j], M[d-j]) = ?$$

is necessary (for $j = 2, ..., d-2$), in analogy to (2), and also a similar decomposition of the mixed functionals

$$V_{m_1,\ldots,m_k}(K_1,\ldots,K_k, M[d-j]) = ?.$$

4. Flag Representations

The following integral representations were recently obtained in **Hug-Rataj-W. (2013, 2017)**.

Theorem 2. (a) *There is a measurable function $f_{j,d-j}$ such that for all K, M (in suitable general position),*

$$V(K[j], M[d-j]) = \int_{F(d,d-j+1)} \int_{F(d,j+1)} f_{j,d-j}(u_1, L_1, u_2, L_2) \times \psi_j(K_1, d(u_1, L_1)) \psi_{d-j}(M, d(u_2, L_2)).$$

(b) *There is a measurable function $g_{m_1,\ldots,m_k}$ such that for all $K_1, \ldots, K_k, M$ (in suitable general position),*

$$V_{m_1,\ldots,m_k}(K_1, \ldots, K_k, M[d-j]) = \int_{F(d,d-j+1)} \int_{F(d,m_k+1)} \cdots \int_{F(d,m_1+1)} g_{m_1,\ldots,m_k}(u_1, L_1, \ldots, u_k, L_k, u, L) \times \psi_{m_1}(K_1, d(u_1, L_1)) \cdots \psi_{m_k}(K_k, d(u_1, L_1)) \, \psi_{d-j}(M, d(u, L)).$$

Here, $\psi_i(K,\cdot)$ denotes the i-th **flag measure** of K, a finite Borel measure on the space

$$F(d,i+1) = \{(u,U) : U \in G(d,i+1), u \in S^{d-1} \cap U\},$$

$$\psi_i(K,\cdot) = \int_{G(d,i+1)} \mathbf{1}((u,U) \in \cdot) S_i'(K|U,du)dU, i = 1,...,d-1.$$

These flag measures also arise from a local Steiner formula in the space $A(d,i+1)$ of affine $(i+1)$-flats, they have nice properties (translation invariant, weakly continuous and additive in K), and the i-th area measure $S_i(K,\cdot)$ is (proportional to) the image of $\psi_i(K,\cdot)$ under the projection $(u,L) \mapsto u$.

Notice that corresponding integral representations do not hold with the area measures, in general.

Using Theorem 2, one can proceed now recursively:

If the mean flag measures

$$\overline{\psi}_{d-1}(X,\cdot)(= c\overline{S}_{d-1}(X,\cdot)), \overline{\psi}_{d-2}(X,\cdot), \dots, \overline{\psi}_{j+1}(X,\cdot)$$

are determined by the mean values

$$\overline{V}_d(Z), \overline{V}(Z[d-1], M), \dots, \overline{V}(Z[j+1], M[d-j-1]),$$

then Theorem 2 shows that the mean value $\overline{V}(Z[j], M[d-j])$ determines $\overline{V}(X[j], M[d-j])$.

Thus, the second challenge is to show that

$$\begin{aligned} \overline{V}(X[j], M[d-j]) &= \int_{F(d,j+1)} \int_{F(d,d-j+1)} f_{j,d-j}(u_1, L_1, u_2, L_2) \\ &\quad \times \psi_{d-j}(M, d(u_2, L_2)) \overline{\psi}_j(X, d(u_1, L_1)) \end{aligned}$$

(where M varies in $\mathcal{K}^d$) determines the measure $\overline{\psi}_j(X,\cdot)$.

This seems to require a (complicated) functional analytic result on the flag space $F(d, j+1)$, but fortunately there is a different approach:

The functional $K \mapsto V(K[j], M[d-j])$ is in the space $\mathbf{Val}_j$ of j-homogeneous, translation invariant, continuous and additive functionals (**valuations**). Confirming a conjecture of McMullen, **Alesker (2001)** has shown that every $\varphi \in \mathbf{Val}_j$ is the limit of finite linear combinations of mixed volumes $V(\cdot[j], M[d-j]), M \in \mathcal{K}^d$.

Therefore, the values $\overline{V}(X[j], M[d-j]), M \in \mathcal{K}^d$, determine all mean values

$$\overline{\varphi}(X) = \gamma \int \varphi(K)\mathbb{Q}(dK), \quad \varphi \in \mathbf{Val}_j.$$

For a continuous function f on $F(d, j+1)$, we have

$$\varphi_f : K \mapsto \int_{F(d,j+1)} f(u,L)\psi_j(K, d(u,L)) \in \mathbf{Val}_j.$$

Hence

$$\int_{F(d,j+1)} f(u,L)\overline{\psi}_j(X, d(u,L))$$

is determined for all f, which yields $\overline{\psi}_j(X, \cdot)$.

At the end of the recursion, all mean flag measures

$$\overline{\psi}_{d-1}(X,\cdot),\ldots,\overline{\psi}_{1}(X,\cdot)$$

are determined, which (by Theorem 2) gives us all mixed expressions in

$$\overline{V}_0(Z) = \mathrm{e}^{-\overline{V}_d(X)}\left(\overline{V}_0(X) - \sum_{k=2}^{d}\frac{(-1)^k}{k!}\right. \\ \left. \times \sum_{\substack{m_1,\ldots,m_k=1\\ m_1+\cdots+m_k=(k-1)d}}^{d-1} \overline{V}_{m_1,\ldots,m_k}(X,\ldots,X)\right).$$

Thus, $\overline{V}_0(Z)$ determines $\overline{V}_0(X)=\gamma$.

Some Remarks

• The results also hold for Boolean models with polyconvex grains.

• Apart from the intensity γ, we also get the mean flag measures $\int \psi_j(K,\cdot)\mathbb{Q}(dK)$, $j = 1,...,d-1$, (and the mean area measures $\int S_j(K,\cdot)\mathbb{Q}(dK), j = 1,...,d-1$).

• If the grains are multiples ηK_0 of a fixed shape K_0 (η a RV), we obtain the first d moments of the distribution of η.

• The approach can also be used for non-stationary Boolean models Z.

References

S. Alesker, Description of translation invariant valuations on convex sets with solution to McMullen's conjecture. *Geom. Funct. Anal.* **11** (2001), 244–272.

D. Hug, J. Rataj, W. Weil, A product integral representation of mixed volumes of two convex bodies. *Adv. Geom.* **13**, 633–662 (2013).

D. Hug, J. Rataj, W. Weil, A flag representation of mixed volumes and mixed functionals of convex bodies. *arXiv 1705.04816*, submitted (2017).

W. Weil, Densities of mixed volumes for Boolean models. *Adv. Appl. Prob. (SGSA)* **33**, 39–60 (2001).