Analysis of image texture heterogeneity using anisotropic multifractional Brownian fields

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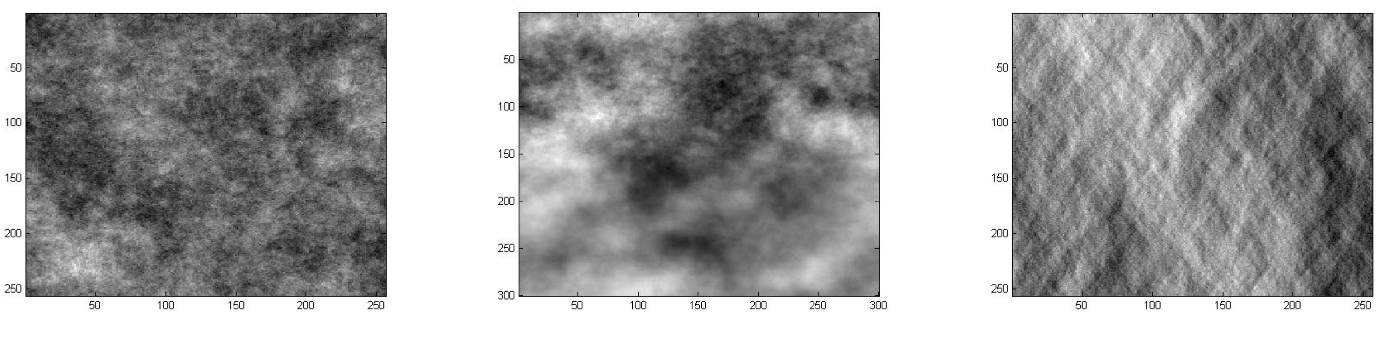


(a) (b) (c)

Figure 1: Homogeneity and heterogeneity of textures of images. (a) Realization of a (isotropic, homogeneity) FBF with constant Hurst function $H = 0.2$, (b) realization of a (anisotropic, homogeneity) AFBF with a Hurst function $\beta = 0.2$ is constant on a section $[-\alpha, \alpha]$ with $\alpha = \arg(\xi) = \pi/4$ and null outside on the uniform grid of size 256×256 of $[0, 1]^2$, (c) realization of a (isotropic, heterogeneity) MBF with vertical gradient of regularity

Abstract

In this poster, we present a model of anisotropic multifractional Brownian field which accounts for local directional properties of image textures. We describe a method based on quadratic variations to estimate main parameters of this model and state an asymptotic normality result about the estimates. We present an application of this method to test the heterogeneity of image textures.

Definition of the model

Locally Anisotropic multifractional Brownian field

$$Z_{\beta,\tau}(x) = \int_{\mathbb{R}^d} (e^{i\langle x,\omega\rangle} - \mathbf{1}_{\{\omega \in B(0,A)\}} P_M(\langle \omega, x\rangle)) \sqrt{f_x(\omega)}\ \widehat{dW}(\omega)$$

where $P_M(u) = 1 - \frac{u^2}{2!} + \cdots + (-1)^M \frac{u^{2M}}{(2M)!}$, and f_x is a spatially heterogeneous density which satisfies

- in the lower frequencies,
$$\forall |\omega| \leq A,\ f_x(\omega) \leq g(\omega),\ \int_{|\omega| \leq A} |\omega|^{2M+2} g(\omega) d\omega < +\infty$$
- in the high frequencies,
$$\forall |\omega| > A,\ \ 0 \leq f_x(\omega) - h_x(\omega) \leq C|\omega|^{-2H_x - d - \gamma}$$
with $h_x(\omega) = |\omega|^{-2\beta_x(\arg(\omega)) - d}\ \tau_x(\arg(\omega))$ and
$H_x = \min\{\beta_x(s), s \in \jmath^{d-1}, \tau_x(s) \neq 0\}$.

Hypothesis for Hurst and topothesy functions: they are two even, positive, and bounded functions defined on the unit sphere $\jmath^{d-1} = \{\omega \in \mathbb{R}^d, |\omega| = 1\}$ of $\mathbb{R}^d$ and

$$\sup_{x,s} |\tau_x(s)| = \overline{\tau} < +\infty,\ \ \tau_x(s) > 0$$
$$\sup_{x,s} |\beta_x(s)| = \overline{H} < +\infty,\ \ \inf_{x,s} |\beta_x(s)| = H > 0$$

$(x, s) \longrightarrow \beta_x(s)$ is locally uniformly and η-holderian $(0 < \eta < 1)$ with rapport to x

Analysis model

Increment

For u, x fixed in $\mathbb{R}^d$, we define

$$V^N_{u,x}[m] = \sum_{k \in [0,L]^d} v[k] Z\left(\frac{m - T_u k + p^N_x}{N}\right)$$

where

- v is the kernel of a filter which annulled the polynomials of degree M. Typical, kernel are of form
$$v[k] = (-1)^{|k|} C^{k_1}_{L_1} \cdots C^{k_d}_{L_d},\ \text{if } k \in [\![1, L_1]\!] \times \cdots \times [\![1, L_d]\!], v[k] = 0 \text{ otherwise},$$
- T_u is the transformation (rotation + scaling)
$$T_u = |u| \begin{pmatrix} \cos(\arg(u)) & -\sin(\arg(u)) \\ \sin(\arg(u)) & \cos(\arg(u)) \end{pmatrix},$$
- $p^N_x = E(Nx)$.

Local quadratic variations

- Quadratic Variations
$$W^N_{u,x} = \frac{1}{N_\epsilon} \sum_{m \in \mathcal{V}^N_\epsilon} (V^N_{u,x}[m])^2$$
on a local neighborhood $\mathcal{V}^N_\epsilon = \{j \in \mathbb{Z}^d, |j/N| \leq \epsilon\}$ of size $N_\epsilon = \#\mathcal{V}^N_\epsilon$.
- Log-variations :
$$Y^N = \left(\log(W^N_{u,x})\right)_{u \in \mathcal{F}, x \in G}.$$

Asymptotic normality

Theorem Asymptotic normality

Z is a LAMBF of order M and order increments $K \geq M + 1$. Then,

$$N^{\frac{d}{2}}(Y^N - \zeta^N) \xrightarrow{(d)} \mathcal{N}(0, \Sigma)$$

where the components of ζ^N are of the form

$$\forall u \in \mathcal{F}, x \in \mathcal{G},\ \zeta^N_{u,x} = \rho_u H_x + \beta^N(x, \varphi_u).$$

with $\beta^N(x, \theta) = \log\left(\frac{1}{N^{2H_x}} \int_0^{+\infty} \int_{E_x} |\hat{v}(R'_\theta \omega)|^2 \delta_x(\theta) |\omega|^{-2H_x - 1}\ d\omega d\theta\right) - d\log(2\pi)$, $\delta_x(s) = \lim_{\rho \to +\infty} h_x(s\rho)\rho^{2H_x + d}$ and $\rho_u = \log(\tilde{\rho}_u)$ with $\tilde{\rho}_u, \varphi_u$ are rescaling factor and rotation angle, resp.

Test of homogeneity

The linear model

Consider terms of Gaussian linear model

$$\forall u = (i, j) \in \mathcal{F}, k \in \mathcal{G}, Y^N_{ijk} = \rho_{ij} H_k + \beta^N_{ki} + \epsilon^N_{ijk}, \tag{1}$$

where,

- k: position, $i \in [\![1, K]\!]$,
- i: orientation, $i \in [\![1, I]\!]$,
- j: scale. $j \in [\![1, J_i]\!]$ and $J = \sum_{i=1}^{I} J_i$,
- ϵ^N_{ijk}: correlated Gaussian variable,
- $\rho_{ij} = \log(|u_{ij}|^2)$.

The test hypotheses

$\mathcal{H}^1_0$: Irregularity and directionality homogeneity

$$\begin{cases} H_k &= H^0,\ \ \forall k \in \mathcal{G} \\ \beta_{ki} &= \beta^0_i,\ \forall k \in \mathcal{G}, i \in [\![1, I]\!]. \end{cases}$$

$\mathcal{H}^2_0$: Irregularity homogeneity

$$H_k = H^0,\ \forall k \in \mathcal{G}$$

Model (1) can also be expressed in a matricial form

$$Y^N = X\theta^N + \epsilon^N,\ \ \epsilon^N \sim \mathcal{N}(0, \Sigma), \tag{2}$$

Testing $\mathcal{H}^1_0$ or $\mathcal{H}^2_0$ amount to check if parameters θ are in a subspace Θ_0

$$\Theta_0 = \{\theta^N \in \Theta; L_0\theta^N = 0\}, \tag{3}$$

for some matrix L_0 of full row-rank $p \leq K(I + 1)$.

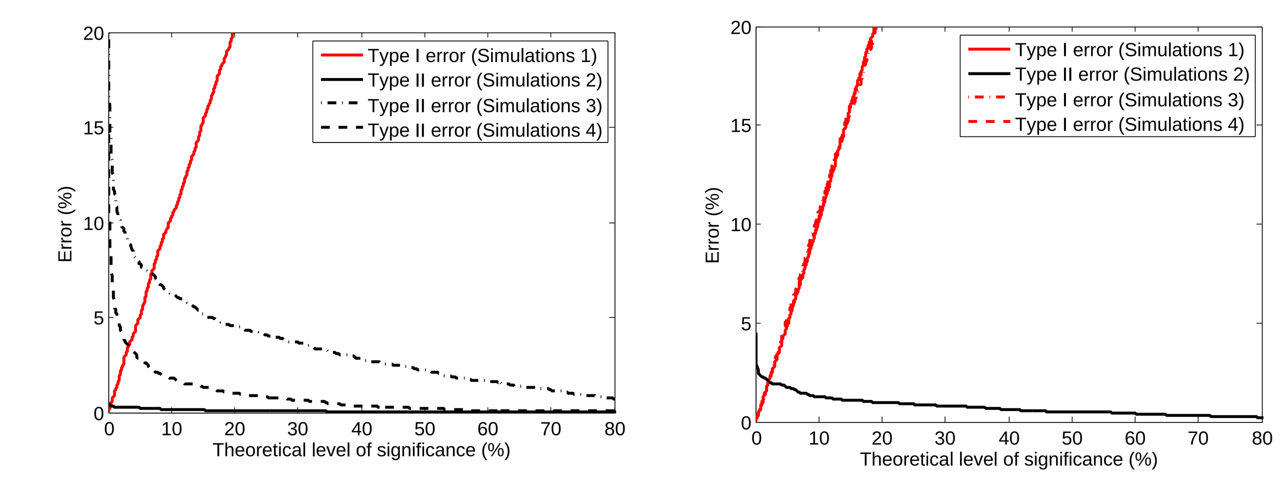
Fisher statistics

$$F^N(\Sigma) = \frac{\hat{\theta}' L'_0 (L_0 (X'\Sigma^{-1}X)^{-1} L'_0)^{-1} L_0 \hat{\theta}}{(Y - X\hat{\theta})'\Sigma^{-1}(Y - X\hat{\theta})} \cdot \frac{KJ - K(1 + I)}{p}, \tag{4}$$

where $\hat{\theta}$ is the generalized least square estimate of parameters θ given by

$$\hat{\theta}^N = P(\Sigma)Y^N \ \text{ with } \ P(\Sigma) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}. \tag{5}$$

Numerical study



(a) $\mathcal{H}^1_0$ (b) $\mathcal{H}^2_0$

Figure 2: Evaluation of the homogeneous tests on 5000 simulated fields with filter (0,2).

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