Intrinsic Volumes of Random Polytopes in Convex Bodies

Nicola Turchi - nicola.turchi@rub.de, joint work with Christoph Thäle and Florian Wespi

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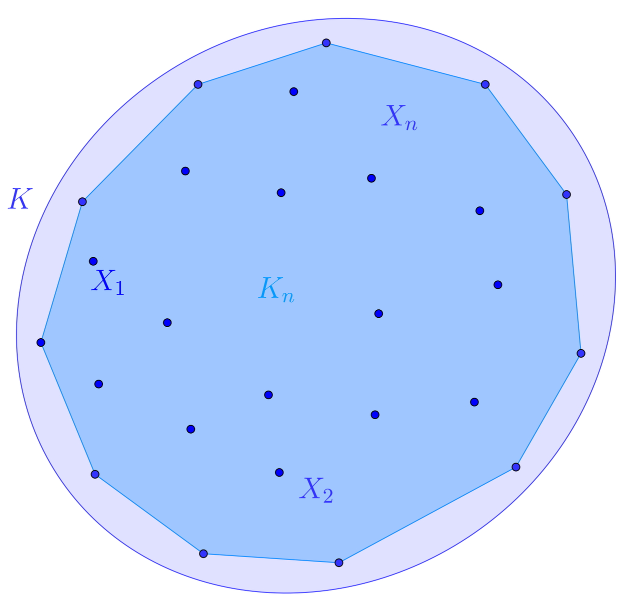
Notation

- $\operatorname{conv}\{x_1,\dots,x_n\}$: **convex hull** of the points $x_1,\dots,x_n$ in $\mathbb{R}^d$.
- $G(d,\ell)$: Grassmannian of all ℓ-dimensional linear subspaces of $\mathbb{R}^d$.
  $\nu(\mathrm{d}L)$: (unique) Haar probability measure on $G(d,\ell)$.
- B_d : euclidean unit ball in $\mathbb{R}^d$.
- $K|L$: orthogonal projection of K onto L.
- $\operatorname{vol}_d(K)$: Lebesgue measure of $K\subseteq\mathbb{R}^d$, $\kappa_d=\operatorname{vol}_d(B_d)$.
- For a Polish space S and $x\in\cup_{k=1}^n S^k$,
  $x^i:=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)$, analogously for x^{ij}.
- For $f\colon\cup_{k=1}^n S^k\to\mathbb{R}$, the **difference operators** are
  $D_if(x):=f(x)-f(x^i)$ and $D_{i,j}f(x):=f(x)-f(x^i)-f(x^j)+f(x^{ij})$.
- $d_W(U_1,U_2)=\sup\{|\mathbf{E}(h(U_1))-\mathbf{E}(h(U_2))|: h\colon\mathbb{R}\to\mathbb{R}\text{ is 1-Lip}\}$ is the Wasserstein distance between two real valued random variables U_1 and U_2.
- $X=(X_1,\dots,X_n)$ is a random vector of elements of S.
  When $X',\widetilde{X}$ are independent copies of X, $Z=(Z_1,\dots,Z_n)$ is a recombination of $\{X,X',\widetilde{X}\}$ if $Z_i\in\{X_i,X_i',\widetilde{X}_i\}$, $i\in\{1,\dots,n\}$.
- $a_n\ll b_n$: $\exists c, c>0$ and $N\in\mathbb{N}$ such that $a_n\le c\,b_n$ whenever $n>N$.

Objective

Obtain quantitative central limit theorems for volumes of random polytopes inscribed in smooth convex bodies.

Settings



- $K\subseteq\mathbb{R}^d$ is a smooth convex body, with strictly positive Gaussian curvature everywhere in ∂K;
- choose n independent random points $X_1,\dots,X_n$ uniformly on K;
- define $K_n:=\operatorname{conv}\{X_1,\dots,X_n\}$.

$\implies$ K_n is a **random polytope** inscribed in the convex body K.

Intrinsic volumes

For $\ell\in\{0,\dots,d\}$, the ℓ-th **intrinsic volume** $V_\ell(K)$ of K can be defined via Kubota's formula,

$$V_\ell(K):=\binom{d}{\ell}\frac{\kappa_d}{\kappa_\ell\,\kappa_{d-\ell}}\int_{G(d,\ell)}\operatorname{vol}_\ell(K|L)\,\nu_\ell(\mathrm{d}L)\,.$$

- They emerge from Minkowski sums of convex bodies:

$$\operatorname{vol}_d(K+tB_d)=\sum_{\ell=0}^{d}V_\ell(K)\operatorname{vol}_{n-\ell}(tB_{n-\ell}),\quad t>0.$$

- Hadwiger's theorem: any continuous valuation $v(K)$ on the class of convex bodies of $\mathbb{R}^d$ which is invariant under rigid motion can be represented as

$$v(K)=\sum_{\ell=0}^{d}c_\ell\,V_\ell(K),$$

  with constants $c_\ell>0$.
- Meaning of $V_\ell(K)$ for some particular values of ℓ:
  - $V_d(K)$ is the ordinary volume,
  - $V_{d-1}(K)$ is half of the surface area,
  - $V_1(K)$ is a constant multiple of the mean width,
  - $V_0(K)$ is the Euler-characteristic of K.

Floating bodies

Consider an hyperplane H such that

$$\operatorname{vol}_d(K\cap H)=t,\quad t>0.$$

Then $K\cap H$ is called a t-cap of K. The **floating body** of K with parameter t is defined by

$$K\setminus K(t)=\bigcup_{K\cap H\text{ is a }t\text{-cap}}K\cap H.$$

Proposition (Bárány, Dalla 1997)

For any $\alpha>0$, there exists $c_\alpha>0$, such that, for $\tau_n:=c_\alpha\frac{\log n}{n}$, it holds

$$\mathbf{P}(K(\tau_n)\subseteq K_n)\ge 1-n^{-\alpha}.$$

Normal approximation bound

$$\gamma_1:=\sup_{(Y,Y',Z,Z')}\mathbf{E}\Big[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\,\mathbf{1}\{D_{1,3}f(Y')\neq 0\}\,(D_2f(Z))^2\,(D_3f(Z'))^2\Big]\,,$$
$$\gamma_2:=\sup_{(Y,Z,Z')}\mathbf{E}\Big[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\,(D_1f(Z))^2\,(D_2f(Z'))^2\Big]\,,$$
$$\gamma_3:=\mathbf{E}\Big[|D_1f(X)|^4\Big]\,,$$
$$\gamma_4:=\mathbf{E}\Big[|D_1f(X)|^3\Big]\,,$$

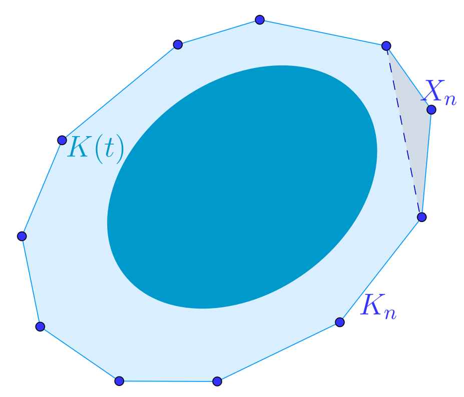
where the suprema in the definitions of γ_1 and γ_2 run over all quadruples or triples of vectors (Y,Y',Z,Z') or (Y,Z,Z') that are recombinations of $\{X,X',\widetilde{X}\}$, respectively. From Stein's method, the following result was proven.

Proposition (Lachièze-Rey, G. Peccati 2016)

Let $W:=f(X_1,\dots,X_n)$ be such that $\mathbf{E}W=0$ and $\mathbf{E}W^2<\infty$. Let N be a standard Gaussian random variable. Then, it holds

$$d_W\left(\frac{W}{\sqrt{\mathbf{Var}\,W}},N\right)\ll\frac{\sqrt{n}}{\mathbf{Var}\ W}\left(\sqrt{n^2\gamma_1}+\sqrt{n\gamma_2}+\sqrt{\gamma_3}\right)+\frac{n}{(\mathbf{Var}\ W)^{\frac{3}{2}}}\gamma_4,$$

Main result



We use the fact that the surface body is contained in the random polytope with high probability, to obtain bounds for $D_i(V_\ell(K_n))$ and the other objects required to apply the normal approximation bound. Finally, we get the following theorem.

Theorem - Central limit theorems for intrinsic volumes

Consider the standardized intrinsic volume

$$W_\ell(K_n):=\frac{V_\ell(K_n)-\mathbf{E}[V_\ell(K_n)]}{\sqrt{\mathbf{Var}[V_\ell(K_n)]}},\qquad \ell\in\{1,\dots,d\}.$$

Then $W_\ell(K_n)$ converges in distribution, as $n\to\infty$, to a standard Gaussian random variable.

Remarks

- The main theorem is actually achieved via a quantitative bound on the Wasserstein distance, namely, it is proven that

$$d_W(W_\ell(K_n),N)\ll n^{-\frac{1}{2}+\frac{1}{n+1}}(\log n)^{3+\frac{2}{n+1}}.$$

  Such rate of convergence is however not optimal, since it was proven in [4] - recently and independently from us - that it holds without logarithmic term.
- For $K=B_d$ the theorem was known. For general K it was known when $\ell=d$.
- We also obtain a quick proof for an asymptotic upper bound on $\mathbf{Var}\,V_\ell(K_n)$, but the optimal bound $n^{-\frac{d+3}{d+1}}\ll\mathbf{Var}\,V_\ell(K_n)\ll n^{-\frac{d+3}{d+1}}$ was already known from [2]. The latter is used in the proof of our theorem.

Further results

A similar approach can be used to study the intrinsic volumes of random polytopes with vertices on the boundary of smooth convex bodies. In particular, combining estimates for the so-called **surface body** with the Efron-Stein jackknife inequality, we obtain lower and upper bounds on the variances of the intrinsic volumes, together with central limit theorems. This is a work in progress jointly with F. Wespi.

References (short list)

[1] **I. Bárány and L. Dalla (1997)**: *Few points to generate a random polytope.* Mathematika 44 44, 325–331.

[2] **I. Bárány, F. Fodor and V. Vigh (2010)**: *Intrinsic volumes of inscribed random polytopes in smooth convex bodies.* Adv. Appl. Probab. 42, 605–619.

[3] **R. Lachièze-Rey and G. Peccati:** *New Berry-Esseen bounds for functionals of binomial point processes.* to appear in Ann. Appl. Probab., (2016+).

[4] **R. Lachièze-Rey, M. Schulte, and J. Yukich**: *Normal approximation for sums of stabilizing functionals.* arXiv: 1702.00726.

This poster is based on the article:

C. Thäle, N. Turchi, F. Wespi: *Random polytopes: variances and central limit theorems for intrinsic volumes.* arXiv:1702.01069