Finding Patterns in Brownian Motion

Hermann Thorisson
University of Iceland

Joint work with Günter Last, Karlsruhe,
Peter Mörters, Bath, and Wenpin Tang, Berkeley

Stochastic Geometry, Stereology and Image Analysis
Marseille / Luminy 18 May 2017

Finding an Extra Head

Consider a two-sided sequence of i.i.d. fair coin tosses.

The Extra Head Problem – Liggett 2002

Can you shift the origin 0 to one of the heads in such a way that you have two independent one-sided fair i.i.d. sequences, one to the left and one to the right of that head?

Note that if you shift the origin to the first head at or after 0 then the sequence to the left of that head will be biased: the distance to the first head to the left will not be geometric, it will be the sum of two independent geometric variables -1. (This is the waiting time paradox.)

Finding an Extra Head

Consider a two-sided sequence of i.i.d. fair coin tosses.

The Extra Head Problem – Liggett 2002

Can you shift the origin 0 to one of the heads in such a way that you have two independent one-sided fair i.i.d. sequences, one to the left and one to the right of that head?

Note that if you shift the origin to the first head at or after 0 then the sequence to the left of that head will be biased: the distance to the first head to the left will not be geometric, it will be the sum of two independent geometric variables -1.

Liggett's solution

If there is a head at 0, do not shift. If there is a tail at 0, shift forward until you have equal number of heads and tails. The origin will then be at a head and it is an extra head.

Palm theory – Balancing allocations

Let $\hat{X}$ be a stationary ergodic r.e. in a space on which $\mathbb{R}$ acts measurably. Let θ_s be the shift by $s \in \mathbb{R}$. Let the r. measures $\hat{\xi}$ and $\hat{\eta}$ be invariant factors of X with the same finite intensity.

Let X and Y be Palm versions of $\hat{X}$ w.r.t. $\hat{\xi}$ and $\hat{\eta}$, respectively. Let ξ and η be the same factors of X as $\hat{\xi}$ and $\hat{\eta}$ are of $\hat{X}$.

An allocation τ is a map of the form $\tau(s) = s + \tau_0(\theta_s X),\ s \in \mathbb{R}$, for some measurable τ_0. It balances ξ and η if $\xi(\tau \in \cdot) = \eta$.

Palm theory – Balancing allocations

Let $\hat{X}$ be a stationary ergodic r.e. in a space on which $\mathbb{R}$ acts measurably. Let θ_s be the shift by $s \in \mathbb{R}$. Let the r. measures $\hat{\xi}$ and $\hat{\eta}$ be invariant factors of X with the same finite intensity.

Let X and Y be Palm versions of $\hat{X}$ w.r.t. $\hat{\xi}$ and $\hat{\eta}$, respectively. Let ξ and η be the same factors of X as $\hat{\xi}$ and $\hat{\eta}$ are of $\hat{X}$.

An allocation τ is a map of the form $\tau(s) = s + \tau_0(\theta_s X)$, $s \in \mathbb{R}$, for some measurable τ_0. It balances ξ and η if $\xi(\tau \in \cdot) = \eta$.

Theorem

$$\theta_{\tau_0(X)} X \overset{D}{=} Y \iff \tau \text{ balances } \xi \text{ and } \eta$$

Palm theory – Balancing allocations

Let $\hat{X}$ be a stationary ergodic r.e. in a space on which $\mathbb{R}$ acts measurably. Let θ_s be the shift by $s \in \mathbb{R}$. Let the r. measures $\hat{\xi}$ and $\hat{\eta}$ be invariant factors of X with the same finite intensity.

Let X and Y be Palm versions of $\hat{X}$ w.r.t. $\hat{\xi}$ and $\hat{\eta}$, respectively. Let ξ and η be the same factors of X as $\hat{\xi}$ and $\hat{\eta}$ are of $\hat{X}$.

An allocation τ is a map of the form $\tau(s) = s + \tau_0(\theta_s X)$, $s \in \mathbb{R}$, for some measurable τ_0. It balances ξ and η if $\xi(\tau \in \cdot) = \eta$.

Theorem

$$\theta_{\tau_0(X)} X \overset{D}{=} Y \iff \tau \text{ balances } \xi \text{ and } \eta$$

Theorem

If ξ is diffuse and ξ and η are mutually singular then τ with $\tau_0(X) := \inf\{t > 0 : \xi([0,t]) \leq \eta([0,t])\}$ balances ξ and η.

Two-sided Brownian motion

Let $B = (B_s)_{s \in \mathbb{R}}$ be a two-sided standard Brownian motion.

This means that $(B_s)_{s \geq 0}$ and $(B_{-s})_{s \geq 0}$ are independent one-sided standard Brownian motions.

In particular, B has value 0 at 0 (that is, $B_0 = 0$ a.s.)

The (diffuse) local time measure ℓ^x at $x \in \mathbb{R}$ can be defined by

$$\ell^x(A) := \lim_{h \to 0} \frac{1}{h} \int_A 1_{\{x \leq B_s \leq x+h\}} ds, \quad A \in \mathcal{B}.$$

A process distributed as $x + B$
is two-sided Brownian with value x at 0.

More generally, if V has distribution ν and is independent of B
then we say that a process distributed as $V + B$
is two-sided Brownian with distribution ν at 0.

Unbiased two-sided Skorohod imbedding of x

Question

Is there a random time T such that $\theta_T B \overset{D}{=} x + B$ for $x \neq 0$?

Unbiased two-sided Skorohod imbedding of x

Question

Is there a random time T such that $\theta_T B \stackrel{D}{=} x + B$ for $x \neq 0$?

Recall the allocation theorem for Palm versions:

If ξ is diffuse and ξ and η are mutually singular then τ with $\tau_0(X) := \inf\{t > 0 \colon \xi([0,t]) \leq \eta([0,t])\}$ balances ξ and η.

Also the following theorem holds:

Brownian motion B is Palm version w.r.t. local time at 0 of the stationary ergodic $\hat{B}$ with σ-finite distribution $\int_{\mathbb{R}} \mathbb{P}(x + B \in \cdot)dx$.

From this we obtain:

Theorem

For $x \in \mathbb{R}$ define $T^x = \inf\{t > 0 \colon \ell^0([0,t]) = \ell^x([0,t])\}$.

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0.

Unbiased two-sided Skorohod imbedding of x

Question

Is there a random time T such that $\theta_T B \overset{D}{=} x + B$ for $x \neq 0$?

Theorem

For $x \in \mathbb{R}$ define $T^x = \inf\{t > 0 \colon \ell^0([0,t]) = \ell^x([0,t])\}$.

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0.

Unbiased two-sided Skorohod imbedding of ν

Question from the previous slide:

Is there a T such that $\theta_T B \stackrel{D}{=} x + B$ for $x \neq 0$?

Theorem from the previous slide:

For $x \in \mathbb{R}$ define $T^x = \inf\{t > 0 \colon \ell^0([0, t]) = \ell^x([0, t])\}$.

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0.

Question

Let $\nu \neq \delta_0$ be a probability measure on $\mathbb{R}$. Is there a T such that $\theta_T B$ is two-sided Brownian with distribution ν at 0 ?

Unbiased two-sided Skorohod imbedding of ν

Question from the previous slide:

Is there a T such that $\theta_T B \stackrel{D}{=} x + B$ for $x \neq 0$?

Theorem from the previous slide:

For $x \in \mathbb{R}$ define $T^x = \inf\{t > 0 \colon \ell^0([0,t]) = \ell^x([0,t])\}$.

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0.

Question

Let $\nu \neq \delta_0$ be a probability measure on $\mathbb{R}$. Is there a T such that $\theta_T B$ is two-sided Brownian with distribution ν at 0 ?

Theorem

Define the local time at ν by $\ell^\nu = \int \ell^x \, \nu(dx)$ and set

$$T^\nu := \inf\{t > 0 \colon \ell^0([0,t]) = \ell^\nu([0,t])\}.$$

If $\nu\{0\} = 0$ then $\theta_{T^\nu} B$ is two-sided Brownian with distribution ν at 0

Finding an Extra Excursion

Recall: the countable excursions from 0 don't form a sequence.

Let ν be the σ-finite Itô excursion law. Let A be a set of excursion paths such that $0 < \nu(A) < \infty$, for instance

excursions with hight $> h > 0$ or with length $> d > 0$.

These excursions form a two-sided sequence.

Let N_A be the simple point process formed by the left end points of these excursions and normalised to have intensity 1. Put

$$T = \inf\{t > 0 \colon \ell^0([0, t]) \leq N_A([0, t])\}.$$

Theorem: If B is two-sided standard Brownian then $\theta_T B$

is standard Brownian in $(-\infty, 0]$,

continues independently in $[0, \infty)$ with a typical excursion of type A, that is, an excursion distributed according to $\nu(\cdot \mid A)$,

and then proceeds independently as standard Brownian.

Finding a Brownian Bridge

The Slepian process $(B_{s+1} - B_s)_{s \in \mathbb{R}}$ is stationary ergodic.
This process has a local-time-at-zero, denote it η.
Set $X_s = (B_{s+u} - B_s)_{0 \le u \le 1}$ and $X = (X_s)_{s \in \mathbb{R}}$.
Let Y be Palm version of X w.r.t. η.
Then (Pitman), Y_0 is a Brownian bridge.

Let ξ be Lebesgue measure, $\xi = \lambda$.
Since X is stationary, X is Palm version of itself w.r.t. λ.

The measures λ and η are diffuse and mutually singular.

Set $T = \inf\{t > 0 : \eta([0, t]) = t\}$ to obtain $\theta_T X \stackrel{D}{=} X'$.

Thus $X_T \stackrel{D}{=} Y_0$, that is, $(B_{T+u} - B_T)_{0 \le u \le 1}$ is a Brownian bridge.

Invariant transports of stationary random measures and mass-stationarity

Annals of Probability 2009

Günter Last and HTh

Unbiased shifts of Brownian motion

Annals of Probability 2014

Günter Last, Peter Mörters and HTh

The Slepian zero set, and Brownian bridge embedded in Brownian motion by a spacetime shift

Electronic Journal of Probability 2015

Jim Pitman and Wenpin Tang

Transporting random measures on the line and embedding excursions into Brownian motion

Annales de l'Institut Henri Poincare, to appear

Günter Last, Wenpin Tang and HTh