Topological reconstruction of r-regular sets

Helene Svane (helenesvane@math.au.dk) - joint with Andrew du Plessis
CSGB, Aarhus University

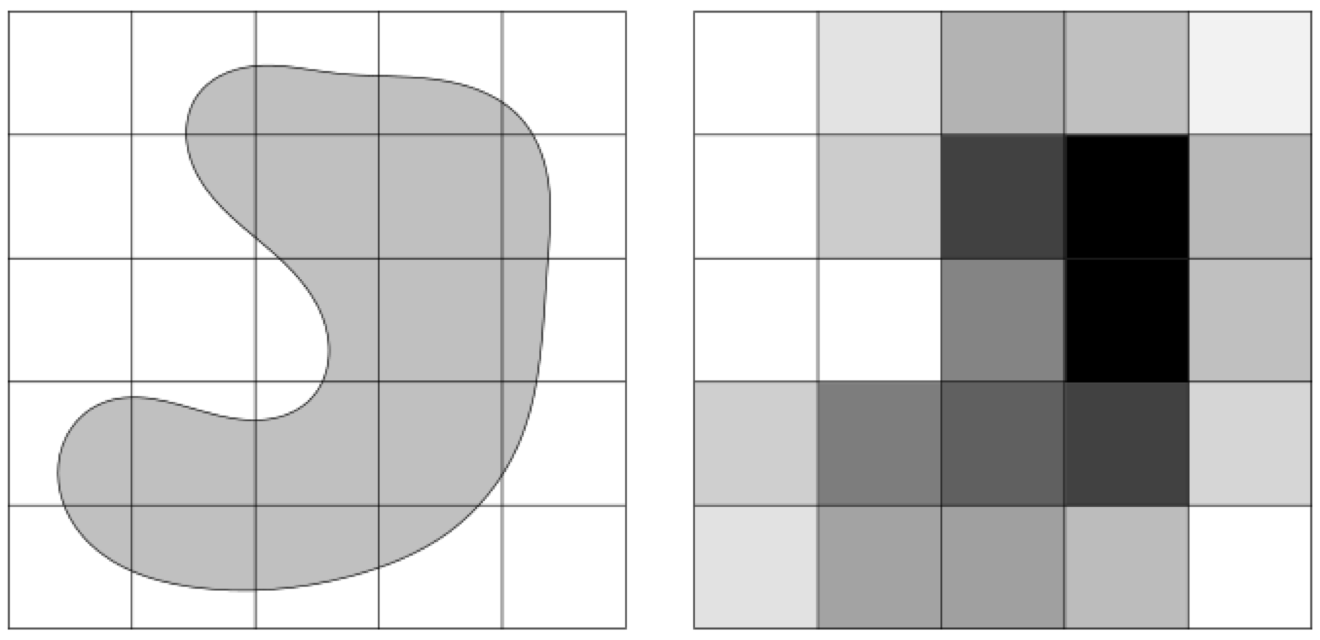
Digitisation of r-regular sets

In many fields of science, interesting 3-dimensional objects are studied using images, i.e. digitisations, of them. In our project, Andrew du Plessis and I have worked on deriving a method for reconstructing the topology of an object from a digitisation of the object. As our input data, we have used the following kind of grey-scale digitisation:

Definition Let $X \subseteq \mathbb{R}^3$ be a subset and $(d\mathbb{Z})^3 \subseteq \mathbb{R}^3$ a lattice. We make a digital reconstruction of X by $(d\mathbb{Z})^3$ in the following way: To each lattice cube C, we assign a number λ corresponding to the intensity of X in C, i.e. the quantity

$$\lambda = \frac{\mathrm{vol}(X \cap C)}{d^3} \in [0,1],$$

where 'vol' denotes the volume. We can now think of the digitisation of X as a collection of grey-scale lattice cubes, each coloured a shade of grey corresponding to the value of λ, so that the cubes that have $\lambda = 0$ will be white, and the cubes that have $\lambda = 1$ will be black.
Let $V(X)$ (or sometimes just V) denote the black cubes of this digitisation of X. We call V the (black) voxel reconstruction of X.



Definition Let $r > 0$. A closed set $X \subseteq \mathbb{R}^n$ is said to be r-regular if the following condition holds:
For each $x \in \partial X$ there exists two r-balls $B_r(x_b) \subseteq X$ and $B_r(x_w) \subseteq X^C$ such that $\overline{B_r(x_b)} \cap \overline{B_r(x_w)} = \{x\}$.
Notice that r-regularity of X implies that both $\mathrm{Reach}(X) \geq r$ and $\mathrm{Reach}(\overline{X}^C) \geq r$.

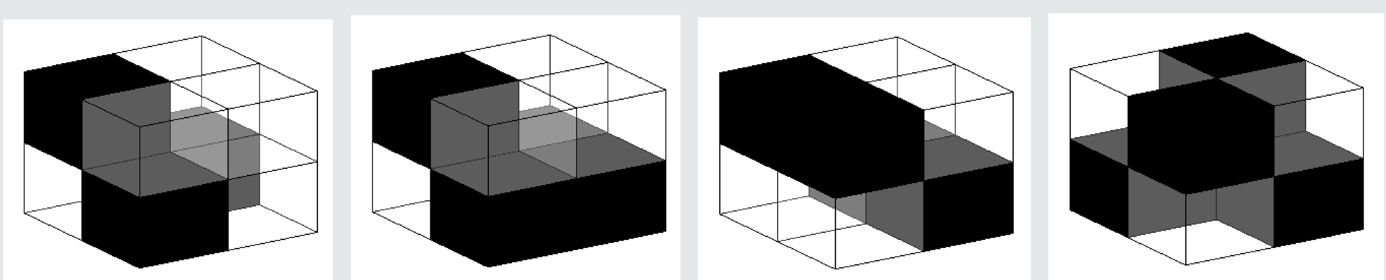
Our goal was now to reconstruct the topology of an r-regular set from the black voxel reconstruction $V(X)$.

Theorem It is possible to reconstruct the topology of an arbitrary r-regular set X from the reconstruction $V(X)$ by a lattice $(d\mathbb{Z})^3$ if $d\sqrt{3} < r$.

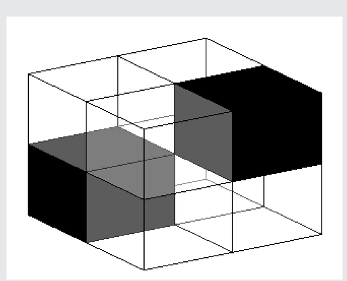
Strategy

Earlier work by du Plessis and Tang Christensen [STC] shows that we can reconstruct the topology of an r-regular object X from its voxel reconstruction V, if we know that certain configurations of black and non-black voxels do not occur in the voxel reconstruction of the set X. As we would like to copy their approach, we showed that this is indeed the case.

Theorem Assume $r > d\sqrt{3}$. In the digital reconstruction V of an r-regular set by a lattice $(d\mathbb{Z})^3$, neither of the following configurations of black and non-black configurations can occur, and nor can their inverses (i.e. the configurations with black and non-black voxels switched):

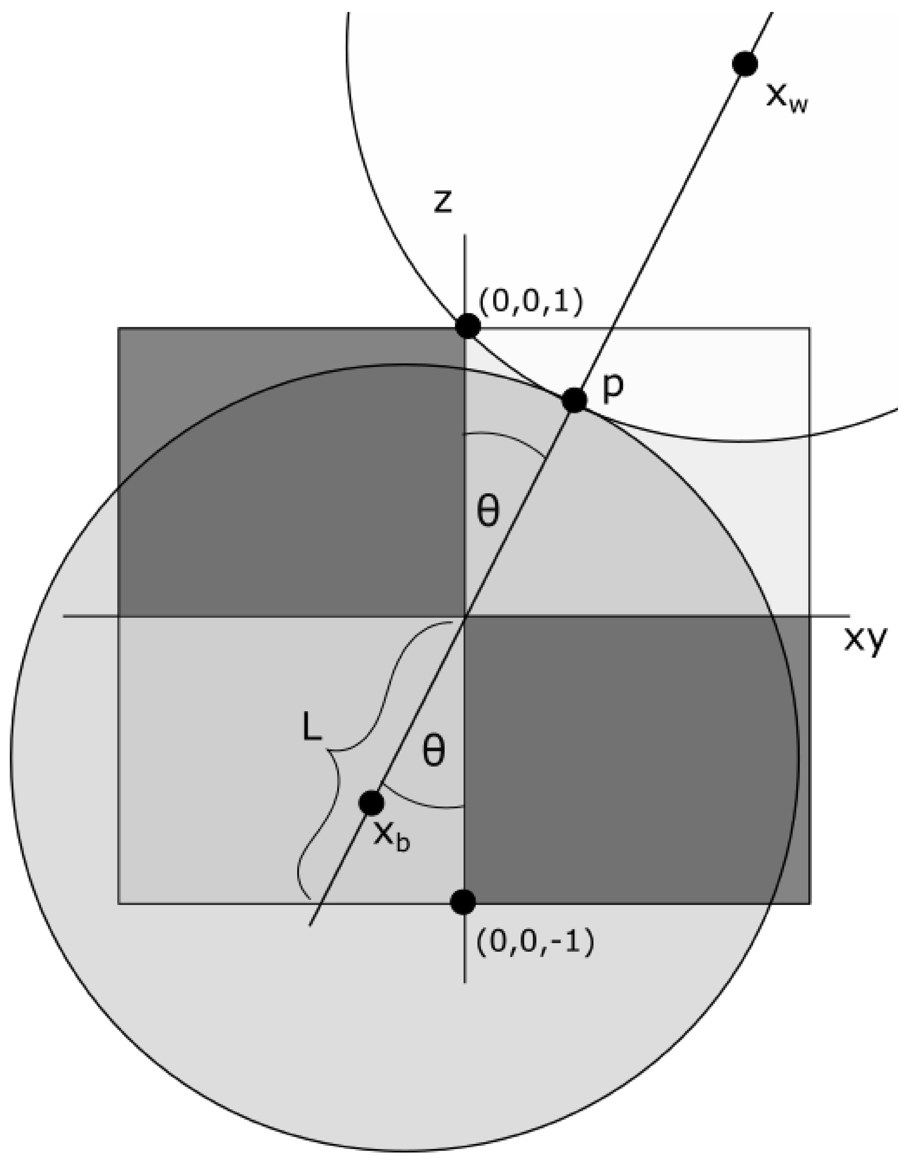


Furthermore, the following configuration cannot occur:



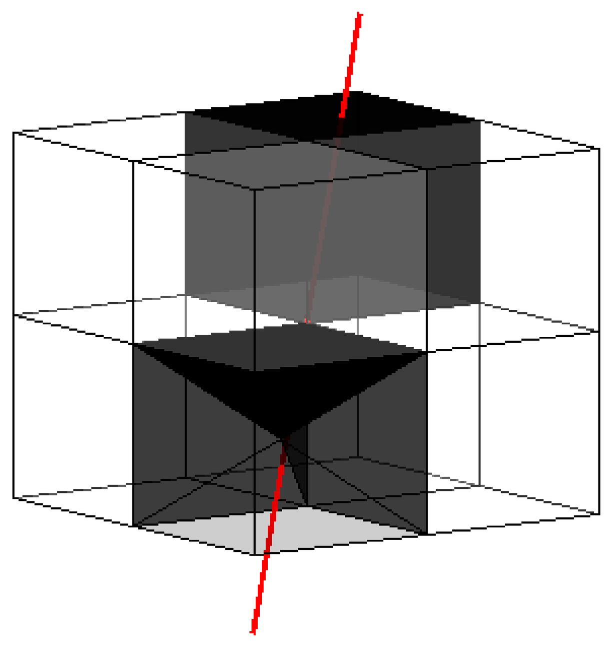
Outline of part of the proof

Consider the bottom configuration in the previous column. To show that this cannot occur in the voxel reconstruction of an r-regular object, the idea is the following:
Let p be the boundary point of ∂X that is closest to the center o of the cube. Then the distance $d(p,o)$ is less than $\sqrt{3}$. There are now two cases: Either the line from p to o passes through one of the black voxels in the configuration, or it does not. If it does not, consider the section in the figure below:



Since p is a boundary point, there are two balls $B_r(x_b) \subseteq X$ and $B_r(x_w) \subseteq X^C$ such that $B_r(x_b) \cap B_r(x_w) = \{p\}$. Both centers x_b and x_w lie on the line through o and p. The idea is to argue that the center x_b of the black ball must belong to a non-black voxel. This will give a contradiction, since it would imply that the entire non-black voxel would be contained in the black ball $B_r(x_b)$.
To show that x_b belongs to a non-black voxel, one calculates how big the distance $d(p,o) + L$ can get and concludes that it is always greater than r.

Suppose in stead that the line from p to o passes one of the black voxels. We argue that the black point x_b will lie so close to one of the non-black voxels that this voxel will still be contained in the black ball $B_r(x_b)$.



Suppose we attach a pyramid of height $\frac{d}{2}$ to each of the non-black voxels neighbouring the black voxel, as is done in the figure above. A calculation shows that the point x_b belongs to one of these three pyramids. Then the corresponding non-black voxel will be contained in $B_r(x_b)$, which gives a contradiction.

Further work

In the theorem, we throw away a lot of information by considering voxels as being either black or non-black. The hope is, however, that by taking grey-scale voxels into account, we can reconstruct not only the topology of the set X, but also the geometry.
A paper on the work presented on this poster is in preparation.

References

[STC] Christensen, Sabrina Tang. *Reconstruction of Topology and Geometry from Digitisations* Ph.D-thesis, 2016.