Anisotropy of Hölder Gaussian random fields: characterization, estimation, and application to image textures.

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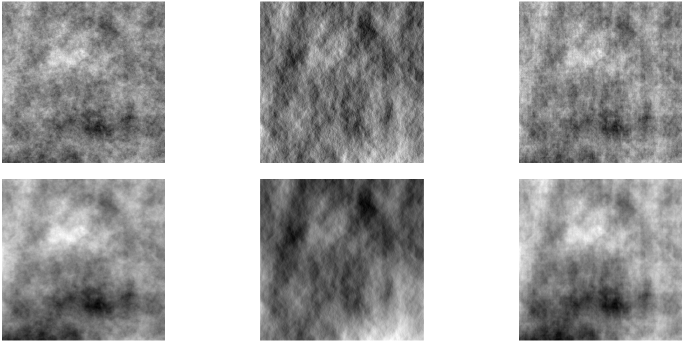
Context and goal



- Context: analysis of rough anisotropic textures of images,
- Goal: characterization and estimation of directional properties associated to the field regularity.

Outline of the talk

1. Characterization.
2. Estimation.
3. Application.

Hölder regularity

- A field Z is Hölder of order $H \in (0,1)$ if
$$|Z(x) - Z(y)| \leq A|x-y|^{\alpha}$$
holds a.s. for any $\alpha < H$, but not for $\alpha > H$.
- If Z is **Gaussian** and **stationary** with an autocovariance
$$E(Z(x+h)Z(x)) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle w,h\rangle} f(w)dw,$$
characterized by a spectral density f.
- Then, Z is H-Hölder iff, for any $0 < \alpha < H$ and $H < \beta < 1$, there exist $A, B, C > 0$ s.t. when $|\mathbf{w}| > \mathbf{A}$
(1) $f(w)|w|^{2\alpha+d} \leq B$,
(2) $f(w)|w|^{2\beta+d} \geq C$,
whenever $\arg(\mathbf{w})$ is in a set E_{β} of positive measure.

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Asymptotic topothesy

- Example (anisotropic fractional Brownian field (*)):

$$\forall \ |w| > A, \ g_{\tau,\beta}(w) = \tau(\arg(w))|w|^{-2\eta(\arg(w))-d}$$

is Hölder of order $H = \operatorname{essinf}\{\eta(s), \tau(s) > 0\}$.

- A more generic model: for some $A, \gamma > 0$,

$$|w| > A \Rightarrow 0 \leq f(w) - g_{\tau,\eta}(w) \leq C|w|^{-2H-d-\gamma}.$$

- For such a model, there exists a bounded and non-vanishing function τ^* defined as

$$\tau^*(s) = \lim_{\rho \to +\infty} \rho^{2H+d} f(\rho s)$$

for almost all spectral directions s.

(*) Bonami and Estrade, 2004

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Asymptotic topothesy and regularity

- Since τ^* is bounded, Hölder index $\geq H$.
- $E_0 = \{s, \tau^*(s) > 0\}$ indicates spectral directions where
  - density convergence is at lowest speeds of order ρ^{2H+d},
  - high-frequencies are the largest.
- Due to high-frequencies in these directions, the field regularity is $\leq H$.
- The asymptotic topothesy: quantifies contributions of directional high-frequencies to the field irregularity.

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A non-stationary framework

- Due to the presence of large polynomial trend, the stationarity assumption is not satisfactory.
- We rather assume that the field is intrinsic, meaning that it has only stationary increments of a specified order.
- Gaussian IRF are characterized by generalized covariance C having a spectral representation [Ref. Gelfand & Villenkin, 1964; Matheron 1973].
- Weak integrability condition on the density:

$$\int_{|w|<\epsilon} |w|^{2M+2} f(w) dw < \infty.$$

Image analysis

- Observed image: $Z^N[m] = Z(m/N), m \in [\![1, N]\!]^d$.
- Increments of order $> M$ at different scales and in different orientations:

$$\forall m \in \mathbb{Z}^d, V_u^N[m] = \sum_{k \in \mathbb{Z}^2} v[k] Z^N[m - T_u k],$$

with a kernel v of order $> M$, and a transform T_u (rotation of angle $\arg(u)$ and a rescaling of factor $|u|$).

- Quadratic variations:

$$W_u^N = \frac{1}{N_e} \sum_{m \in \mathcal{E}_N} (V_u^N[m])^2.$$

Asymptotic normality

Theorem (Richard, 2016)

Let $Y_u^N = \log(W_u^N)$ *and* $x_u^N = \log(|u|^2/N)$. *Define* ϵ_u^N *such that*

$$Y_u^N = H \; x_u^N + \log(\beta_{H,\tau^*}(\arg(u))) + \epsilon_u^N,$$

with

$$\beta_{H,\tau^*}(\theta) = \frac{1}{(2\pi)^d} \int_S \tau^*(\varphi) \; \Gamma_{H,v}(\theta - \varphi) \; d\varphi = \tau^* \circledast \Gamma_{H,v}(\theta),$$

and

$$\Gamma_{H,v}(\theta) = \int_{\mathbb{R}^+} |\hat{v}\,(\rho\theta)|^2 \, \rho^{-2H-1} d\rho,$$

Then, as N tends to $+\infty$, *the random vector* $(N^{\frac{d}{2}} \epsilon_u^N)_{u \in \mathcal{F}}$ *tends in distribution to a centered Gaussian vector.*

An inverse problem

Problem 1: For $j \in \mathcal{J}$, let $\tilde{\beta}_j$ be the estimate of $\beta(\theta_j)$ in some indexed directions θ_j, and $\tilde{H}$ an estimate of H. Define a generalized least square criterion

$$C_{\tilde{H},\tilde{\beta}}(\tau) = \sum_{j,k \in \mathcal{J}} \gamma_{j,k} (\tilde{\beta}_j - \Gamma_{\tilde{H},v} \circledast \tau(\theta_j))(\tilde{\beta}_k - \Gamma_{\tilde{H},v} \circledast \tau(\theta_k)),$$

Find τ^* as the function of $L^2([0, 2\pi))$ which minimizes $C_{\tilde{H},\tilde{\beta}}$.

Problem 2: Find τ^* which minimizes a penalized l.s. criterion

$$\tilde{C}_{\tilde{H},\tilde{\beta},\lambda}(\tau) = C_{\tilde{H},\tilde{\beta}}(\tau) + \lambda |\tau - \tau_0|_W^2, \quad (1)$$

where $\lambda > 0$, $\tau_0 = \int_{[0,\pi)} \tau(\theta) d\theta$ and $|\cdot|_W$ a Sobolev norm.

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Implementation

- Expansion in a cos/sin basis:
  $\tau(\theta) = \tau_0 + \sum_{m=1}^{A} \tau_{1,m} \cos(2m\theta) + \tau_{2,m} \sin(2m\theta)$.
- Discretized criterion:

$$\tilde{C}^{A}_{\tilde{H},\tilde{\beta},\lambda}(\tau) = |L\tau - \tilde{\beta}|^2_\Gamma + \lambda\, \tau^T R \tau$$

- Expressions of relative bias and standard deviation:

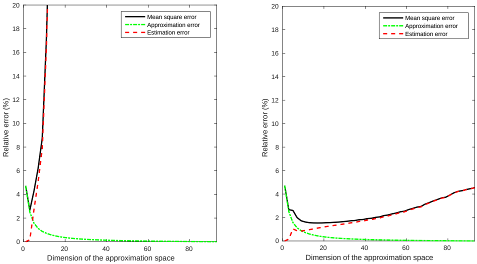
$$\mathrm{rBIAS} = \frac{|\mathbb{E}(\tilde{\tau}^*_\lambda) - \tau^*|}{|\tau^*|} \leq \frac{\lambda\kappa|R|}{\lambda + \nu_+},$$

$$\mathrm{rSTD} = \frac{\sqrt{\mathrm{trace}(\mathrm{V}(\tilde{\tau}^*_\lambda))}}{|\tau^*|} \leq \frac{\kappa\nu_+\sqrt{\nu_-}\,\sqrt{t}}{\beta'\Gamma\beta\,(\lambda + \nu_+)}.$$

  where ν_+, ν_-, κ, t are the largest and lowest eigenvalues, the conditioning number and the trace of $(L'\Gamma L)^{-1}$, respectively.
- The RMSE is minimal for

$$\lambda^* = \frac{\nu_+\nu_-\, t}{|R|^2\ \beta'\Gamma\beta}.$$

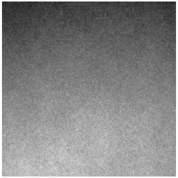
Numerical study



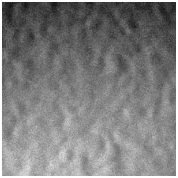
(a) (b)

Figure: Errors obtained (a) without and (b) with penalization.

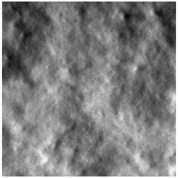
Data: 10000 simulations of anisotropic fractional Brownian fields with uniformly sampled Hurst index.



Glossy



Semi-glossy



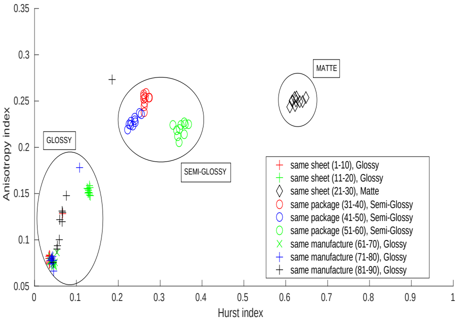
Matte

- Texture of papers: critical feature for conservators, artists, and manufacturer.
- Automated paper classification.
- Collection of raking-light photomicrographs (Paul Messier, conservator in MoMA, NY).

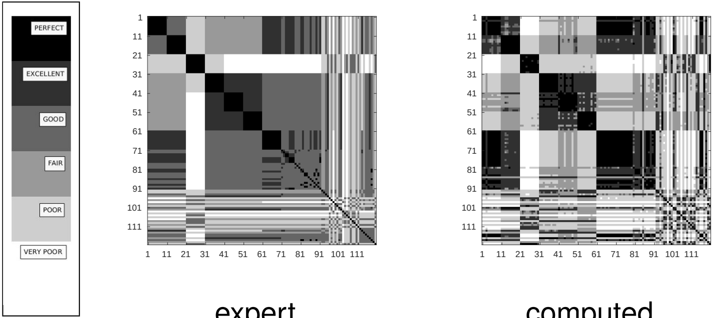
Paper classification

Two classification features: estimates of the Hurst index and an anisotropy index defined as

$$I = \sqrt{\int_{[0,\pi)} \left(\tau^*(s) - \int_{[0,\pi)} \tau^*(u)du \right)^2 ds}.$$



Comparison of affinity matrices



expert computed

- In brief,
  - asymptotic topothesy: a spectral characterization of directional properties associated to the Holder regularity of Gaussian fields,
  - estimation of this function based on quadratic variations of increments and their asymptotic properties.
  - classification of photographic paper textures.
- Perspectives :
  - a partial answer to the issue of the estimation of the topothesy function of anisotropic fractional Brownian field,
  - information about field covariance structure that can be used to deal other image processing tasks (separation trend/texture, examplar-based simulation, inpainting,...).

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