VOLUMES OF CONVEX HULLS OF $n \leq d+1$ POINTS IN d-DIMENSIONAL CONVEX BODIES

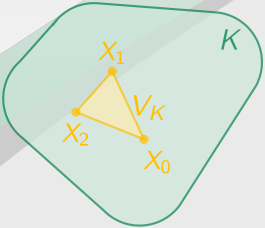
BENJAMIN REICHENWALLNER

May 16, 2017

SETTING

- $\mathcal{K}^d := \{K \subseteq \mathbb{R}^d : K \ d\text{-dimensional convex body}\}$.
- $K \in \mathcal{K}^d$, $d \in \mathbb{N}$, $X_0, \ldots, X_d \in K$ independent random points:

$$V_K := \operatorname{vol} \operatorname{conv}(X_0, \ldots, X_d).$$



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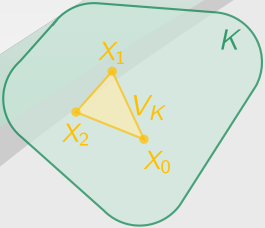

- Consider $\mathbb{E} V_K^k$ for $K \in \mathcal{K}^d, d \in \mathbb{N}$.

KNOWN MOMENTS

- $\mathbb{E}V_K^k$ is known for…
  - **Triangles** (Reed 1974; Alagar 1977),
  - **Convex quadrangles** $(k=1)$ (Herglotz 1933),
  - **Convex polygons** $(k=1)$ (Buchta 1984),
  - **Balls** (Kingman 1969; Miles 1971),
  - **Tetrahedra** $(k=1)$ (Buchta & Reitzner 1993, 2001; Mannion 1994),
  - **Cubes** $(k=1)$ (Zinani 2003).

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  - *Higher dimensions?*

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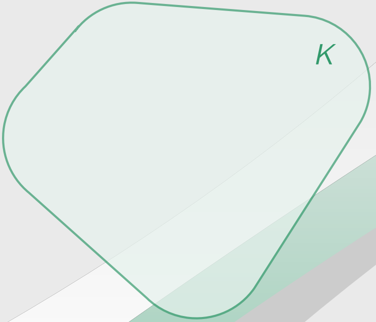

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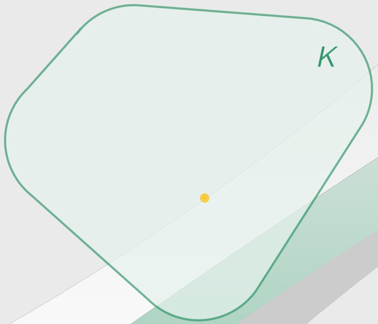

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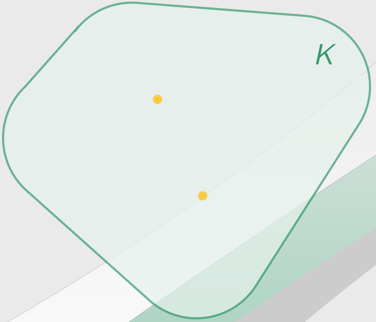

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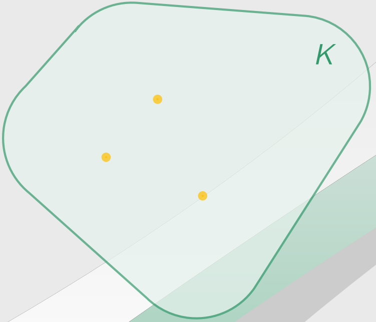

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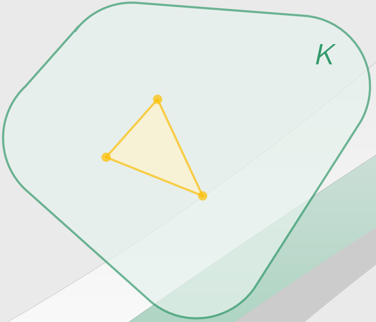

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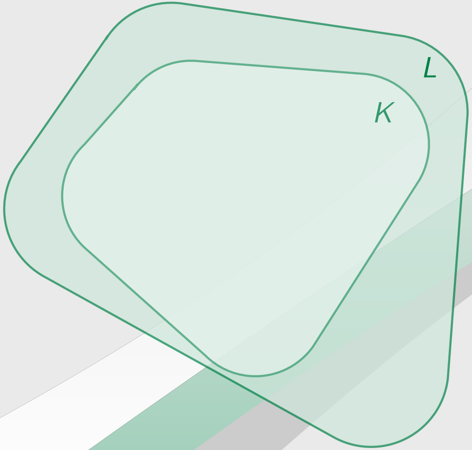

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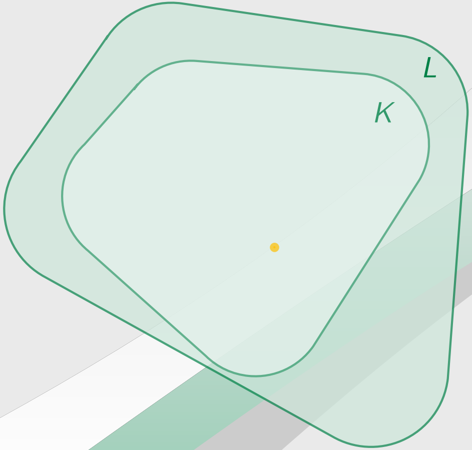

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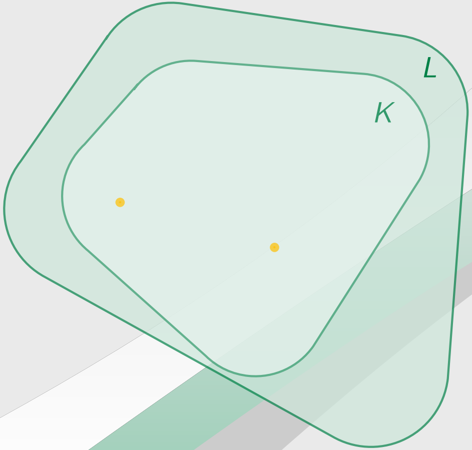

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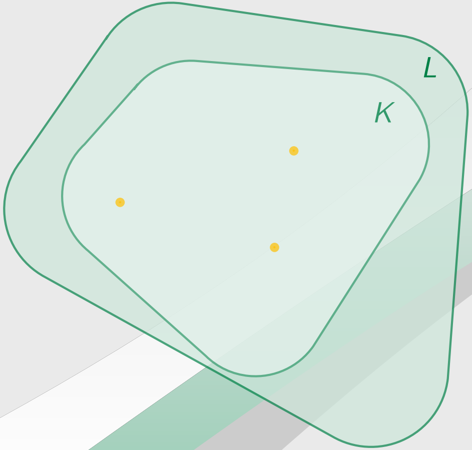

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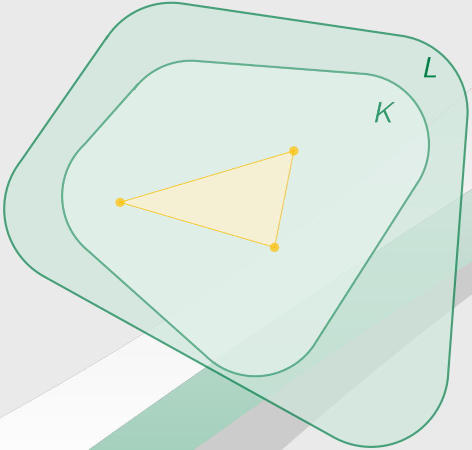

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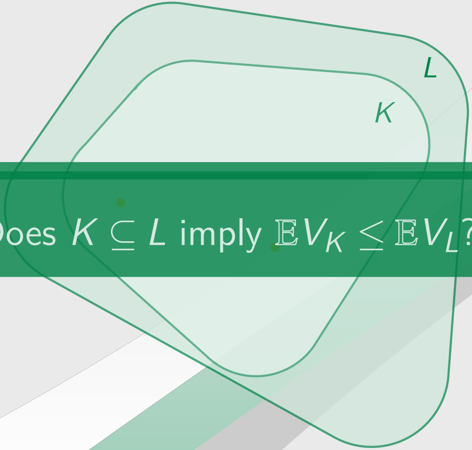

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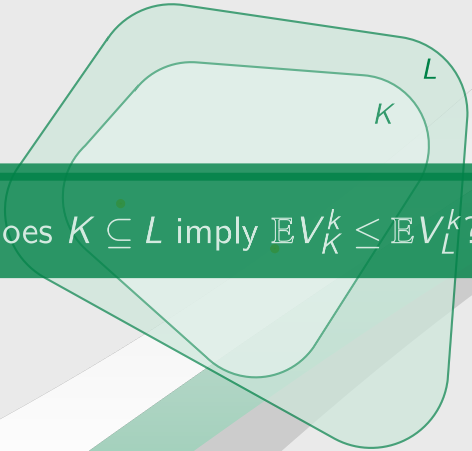

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2012 **L. Rademacher**:

- Monotonicity of $\mathbb{E}V_K$ does **not** hold **in general**.
- Meckes' weak conjecture is equivalent to the **Hyperplane Conjecture**.

MONOTONICITY OF MOMENTS OF V_K

Theorem (Rademacher '12; R. & Reitzner '16; Kunis, R. & Reitzner '17+)

Let $d, k \in \mathbb{N}$, $d \geq 2$. Then there exist two d-dimensional convex bodies K, L satisfying $K \subseteq L$ and $\mathbb{E}V_K^k > \mathbb{E}V_L^k$ unless $d = 2$ and $k \in \{1, 2\}$.

- Rademacher (2012): $d \neq 3$, $k = 1$
- R. & Reitzner (2016): $d \in \{2, 3\}$, $k > 1$
- Kunis, R. & Reitzner (2017+): $d = 3$, $k = 1$

AN EQUIVALENT PROBLEM

Lemma (Rademacher 2012)

Let $d, k \in \mathbb{N}$. The following statements are equivalent:

(i) For each $K, L \in \mathcal{K}^d$, $K \subseteq L$ implies $\mathbb{E}V_K^k \le \mathbb{E}V_L^k$.

(ii) For each $K \in \mathcal{K}^d$ and each $x \in \operatorname{bd} K$,

$$\mathbb{E}V_K^k \le \mathbb{E}V_{K,x}^k.$$

- **Notation:** $K \in \mathcal{K}^d$, $x \in K$, $X_1, \ldots, X_d \in K$ independent random points:

$$V_{K,x} := \operatorname{vol} \operatorname{conv}(x, X_1, \ldots, X_d).$$

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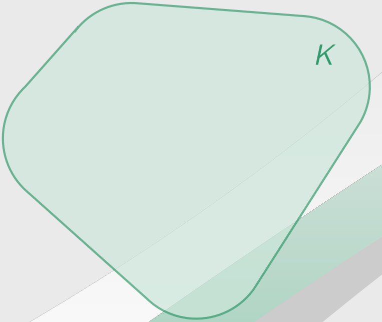

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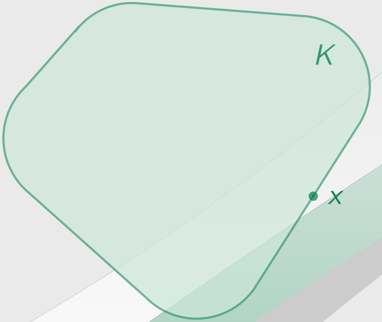

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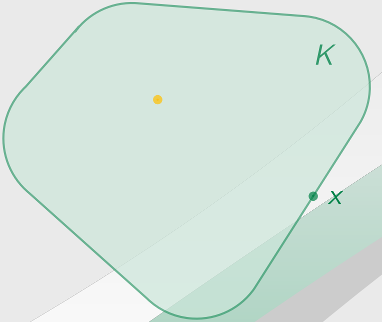

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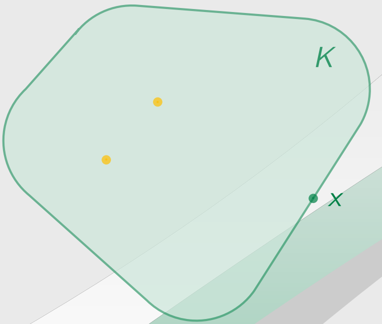

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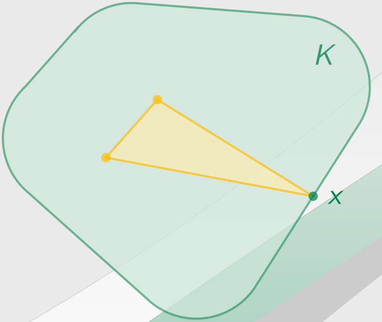

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  - Restricted to $x \in \operatorname{bd} K$, $\frac{\mathbb{E}V_{K,x}^k}{\operatorname{vol} K^k}$ is minimal if and only if K is **half of an ellipsoid centered at** x. (Rademacher 2012)

QUESTION OF INTEREST

- For $d \leq 2, k \in \mathbb{N}$, are there $K \in \mathcal{K}^d, x \in \operatorname{bd} K$ satisfying

$$\mathbb{E}V_K^k > \mathbb{E}V_{K,x}^k?$$

- Two candidates:
  - **Halfball** and the origin o,
  - **Simplex** (and x the midpoint of a facet).

SKETCH OF PROOF

- **Halfball** as a counterexample for:
  - all moments in dimensions $d \geq 4$,
  - the k-th moment in dimension 2 for $k \geq 8$ and
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$$\frac{\mathbb{E}V_T^k}{\operatorname{vol} T^k} = \frac{12}{(k+1)^3(k+2)^3(k+3)(2k+5)} \left(6(k+1)^2 + (k+2)^2 \sum_{i=0}^{k} \binom{k}{i}^{-2}\right).$$

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  - **R. & Reitzner 2016** (c center of an edge of T):

$$\frac{\mathbb{E}V_{T,c}^k}{\operatorname{vol} T^k} = \frac{2^{3-k}}{(k+1)(k+2)^2(k+3)} \left(\sum_{l=1}^{k+1} \binom{k+2}{l}^{-1} + 1 \right).$$

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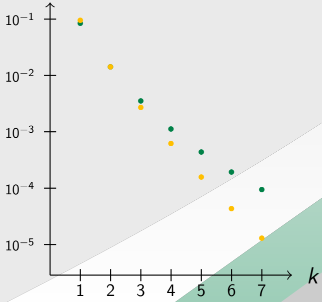

k	$\mathbb{E}V_{T,c}^k$	$\mathbb{E}V_T^k$
1	0.092593...	0.083333...
2	0.013889...	0.013889...
3	0.002667...	0.003444...
4	0.000602...	0.001111...
5	0.000153...	0.000430...
6	0.000043...	0.000190...
7	0.000013...	0.000093...

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  - **Buchta & Reitzner 2001:**

$$\frac{\mathbb{E} V_T}{\mathrm{vol}\, T} = \frac{13}{720} - \frac{\pi^2}{15\,015} = 0.01739\ldots$$

BUCHTA & REITZNER 2001

$$
\begin{aligned}
\frac{\mathbb{E}V_{T[n]}}{\operatorname{vol} T} &= 1 - \frac{2}{n+1} - \frac{3(n-1)n}{4}\left[\frac{1}{(n+1)^3} + \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{1}{(k+3)^3}\right] \\
&\quad - \frac{9(n-1)n}{2} \sum_{\substack{j_1+\ldots+j_5=n-2 \\ k_1+k_2+k_3=4 \\ j_1,\ldots,j_5,k_1,k_2,k_3 \geq 0}} \binom{n-2}{j_1,\ldots,j_4} \binom{4}{k_1,k_2} 2^{k_2}\, 3^{j_2+j_3} \\
&\quad \times B(j_2+2j_3+3j_4+3j_5+k_2+2k_3+1, 3j_1+2j_2+j_3+2k_1+k_2+1) \\
&\quad \times B(n+1, j_5+k_3+1)\ B(2j_1+j_2+k_1+1, j_5+2) \\
&\quad \times {}_3F_2(j_5+k_3, n+1, 2j_1+j_2+k_1+1; j_5+k_3+n+2, 2j_1+j_2+j_5+k_1+3; 1) \\
&\quad + 6(n-1)n \sum_{\substack{j_1+\ldots+j_5=n-2 \\ l_1+l_2=2 \\ l_3+l_4=2 \\ j_1,\ldots,j_5,l_1,l_2,l_3,l_4 \geq 0}} \binom{n-2}{j_1,\ldots,j_4} \binom{2}{l_1} \binom{2}{l_3} 3^{j_2+j_3} \\
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\end{aligned}
$$

TETRAHEDRON IN A TETRAHEDRON T

- With $c = (x_c, y_c, z_c)$ and $X_i = (x_i, y_i, z_i)$:

$$V_{T,c} = \left| \frac{1}{6} \det \begin{pmatrix} x_c & y_c & z_c & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} \right| .$$

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- Even moments of $V_{T,c}$ with vol $T = 1$:

$$\mathbb{E} V_{T,c}^{2k} = \frac{8}{3^{2k-3}} \sum_{\sum_1^{18} k_i = 2k} (-1)^{k'} 3^{k''} \binom{2k}{k_1, \ldots, k_{18}} \prod_{i=1}^{3} \frac{l_i!\, m_i!\, n_i!}{(l_i + m_i + n_i + 3)!} .$$

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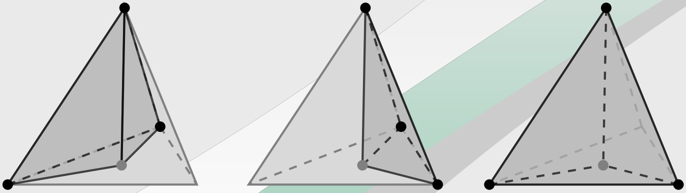

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- $V_{T,c}$ is not larger than $1/3$.
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for some $n \in \mathbb{N}$ such that $P(x) \geq |x|$ for all $x \in [0,1/3]$.

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- Then $\mathbb{E}P(V_{T,c}) \geq \mathbb{E}V_{T,c}$.
- Furthermore, $\mathbb{E}P(V_{T,c}) = \mathbb{E}\sum_{i=0}^{n} a_i V_{T,c}^{2i} = \sum_{i=0}^{n} a_i \mathbb{E}V_{T,c}^{2i}$.

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FIRST IDEA

- Solve the linear program

$$\min_P \sum_{i=0}^{n} a_i \, \mathbb{E} V_{T,c}^{2i} \quad \text{s.t.} \quad P(x) \geq x, \ x \in \left[0, \frac{1}{3}\right].$$

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- For $n = 12$ and $L = 100$: $\mathbb{E}P(V_{T,c}) > 0.01746 > \mathbb{E}V_T$.
- For $n = 13$ and $L = 1\,000$, we get the estimate

$$\mathbb{E}P(V_{T,c}) \approx 0.0173716 < \mathbb{E}V_T = 0.0173982\dots$$

A LEMMA

Lemma

Let $m \in \mathbb{N}$, $n = 2m+1$, and $0 < x_0 < \cdots < x_m$ be given. Then the system of equations

$$P(x_j) = x_j \text{ and } P'(x_j) = 1 \qquad \text{for } j = 0, \cdots, m$$

determines the polynomial $P(x) = \sum_{i=0}^{n} a_i x^{2i}$ uniquely and implies that $P(x) \geq |x|$ for all $x \in \mathbb{R}$.

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  - compute the interpolation nodes with the absolute value function numerically,
  - rationalize these points to $\left\{\frac{1}{83}, \frac{1}{22}, \frac{1}{11}, \frac{2}{15}, \frac{2}{11}, \frac{5}{22}, \frac{4}{15}\right\}$,

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  - solve the interpolation problem exactly:

$$\mathbb{E} V_{T,c} \leq \mathbb{E} P(V_{T,c})$$

$$= \frac{9215716290354844120429638007455369524673678722793816500930077456473045568887048115406728916177539584714665872679760706490830685597598152285959465854826727651}{5302749610595656531552535250244971867247891347471652937573234276503331818475900122041632848466907674944125921368035824590209196687643726617024566886400000000000}$$

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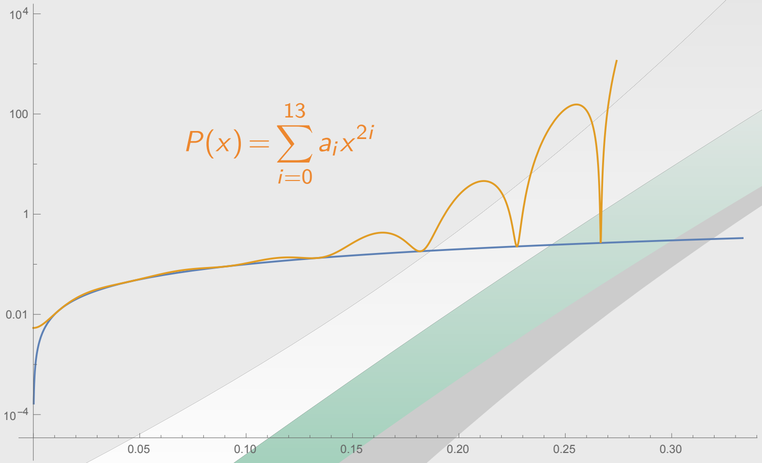
INTERPOLATION POLYNOMIAL



SETTING

- **Notation:** $K \in \mathcal{K}^d$, $2 \leq n \leq d+1$, $X_0, \ldots, X_{n-1} \in K$ independent random points, $x \in K$:

$$V_{K[n]} := \mathrm{vol}_{n-1} \operatorname{conv}(X_0, \ldots, X_{n-1}),$$
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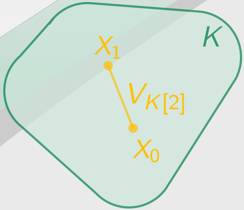

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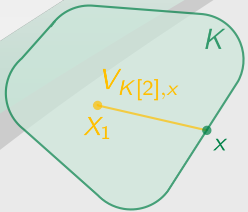

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RESULT ON MONOTONICITY

Theorem (R. 2017+)

Let $d, n, k \in \mathbb{N}$, $d \geq 2$ and $2 \leq n \leq d$. Then there exist two d-dimensional convex bodies K, L satisfying

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Lemma (Rademacher 2012)

Let $d, n, k \in \mathbb{N}$, $2 \leq n \leq d+1$. The following statements are equivalent:

(i) For each $K, L \in \mathcal{K}^d$, $K \subseteq L$ implies $\mathbb{E}V^k_{K[n]} \leq \mathbb{E}V^k_{L[n]}$.

(ii) For each $K \in \mathcal{K}^d$ and each $x \in \operatorname{bd} K$,

$$\mathbb{E}V^k_{K[n]} \leq \mathbb{E}V^k_{K[n],x}.$$

INDUCTION LEMMA

Lemma (R. 2017+)

Let $d, n, k \in \mathbb{N}$ with $2 \leq n \leq d+1$. If there exist $K, L \in \mathcal{K}^d$ satisfying

$$K \subseteq L \quad \text{and} \quad \mathbb{E}V^k_{K[n]} > \mathbb{E}V^k_{L[n]},$$

there also exist $K', L' \in \mathcal{K}^{d+1}$ satisfying

$$K' \subseteq L' \quad \text{and} \quad \mathbb{E}V^k_{K'[n]} > \mathbb{E}V^k_{L'[n]}.$$

SKETCH OF PROOF

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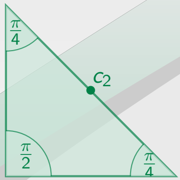

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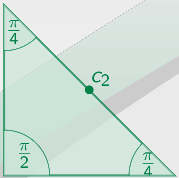

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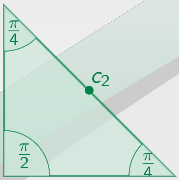

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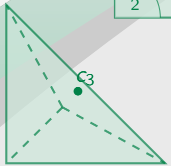

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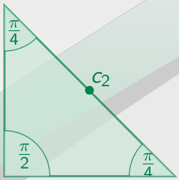

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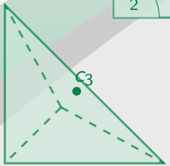
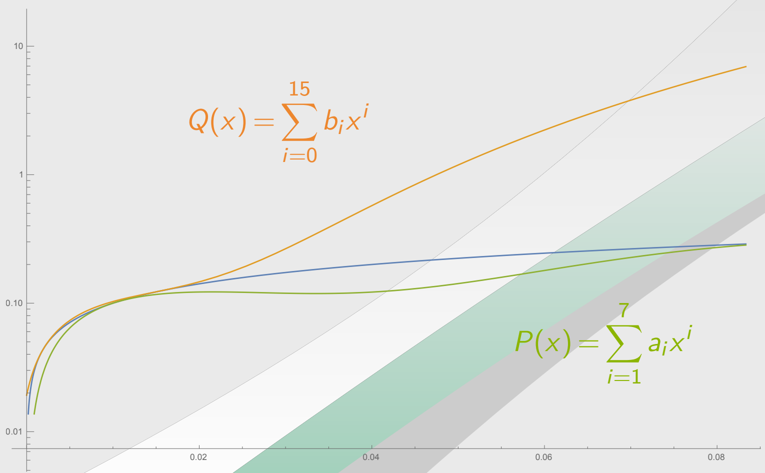


  - $\mathbb{E}V_{T_3[3],c_3} \le \mathbb{E}Q(V_{T_3[3],c_3}) < 0.046942 < \mathbb{E}P(V_{T_3[3]}) \le \mathbb{E}V_{T_3[3]}.$

INTERPOLATION POLYNOMIALS



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