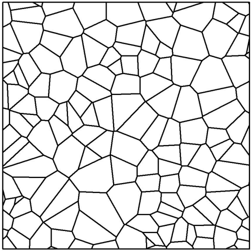
A Mecke-type formula for STIT tessellation processes and some applications

Werner Nagel,

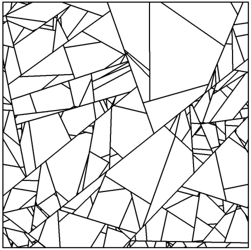
joint work with Christoph Thäle, Viola Weiß
and Linh Ngoc Nguyen

Random tessellations

Three reference models



Poisson-Voronoi



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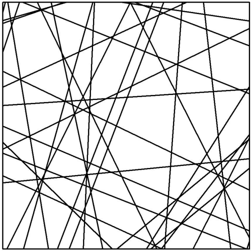

Poisson line

Translation invariant measure on the space of hyperplanes

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- Λ ... translation invariant measure on $(\mathcal{H}, \mathfrak{H})$ (directional distribution not concentrated on a set of hyperplanes parallel to one line)

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$$\Lambda(\mathrm{d}h)\ \mathrm{d}s$$

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- Space-time process $(\Gamma_t,\ t > 0)$ with

$$\Gamma_t = \{(h, s) \in \Gamma :\ s \leq t\}$$

Poisson hyperplane vs. STIT tessellation processes

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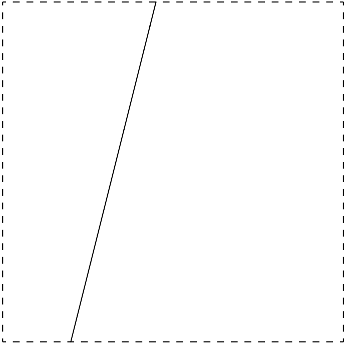

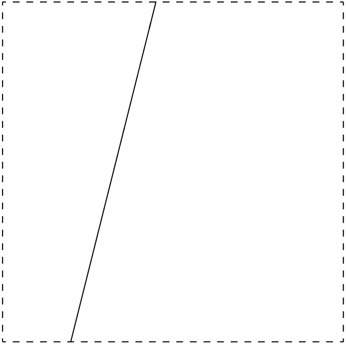

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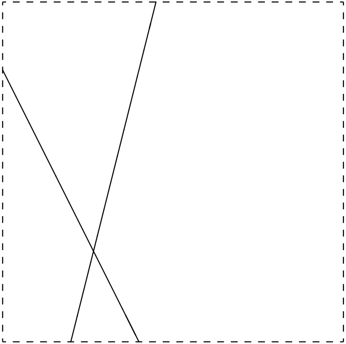

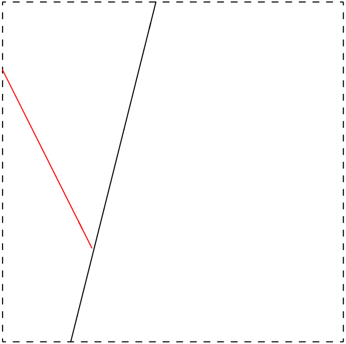

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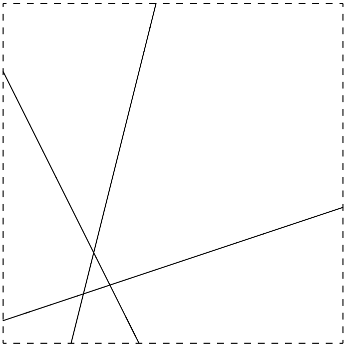

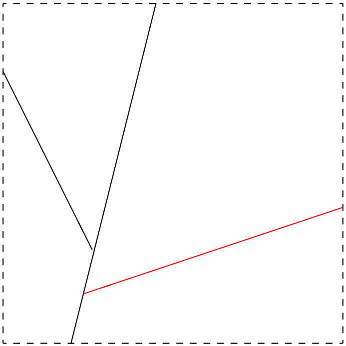

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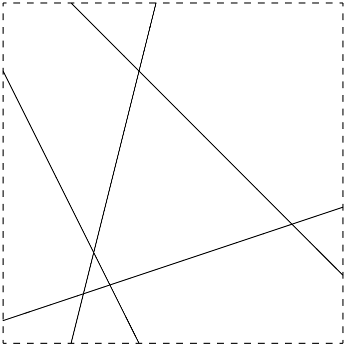

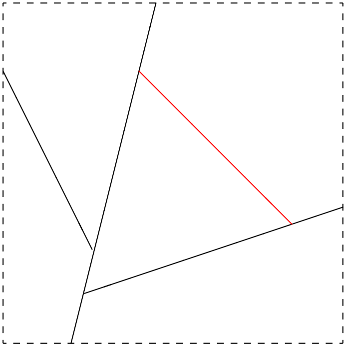

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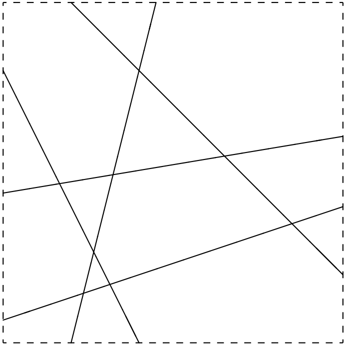

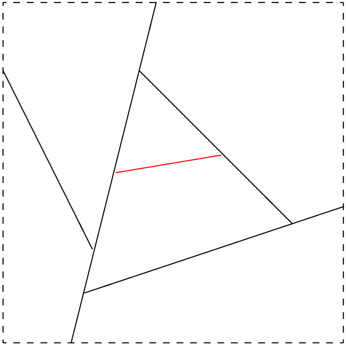

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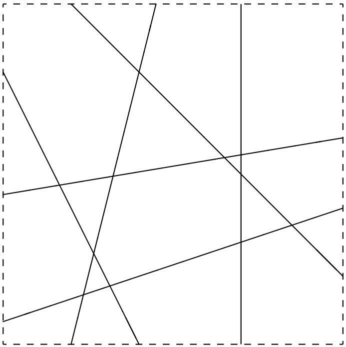

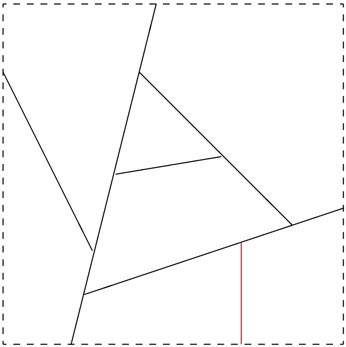

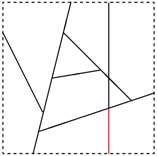

If a hyperplane intersects more than one cell (polytope), $z_1, \ldots, z_k$, say, then select z_j for division with probability

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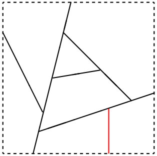

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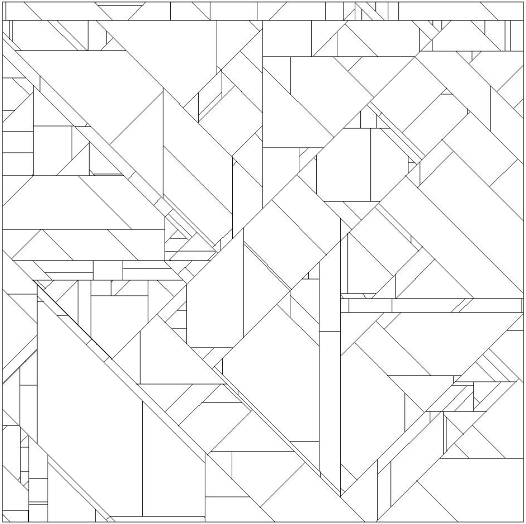
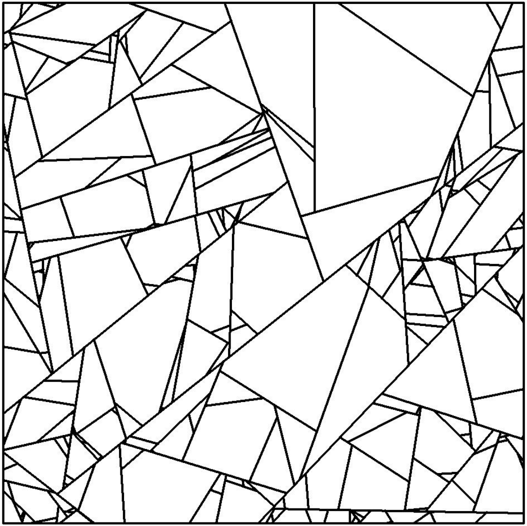
Simulations of STIT tessellations

four directions



Simulations of STIT tessellations

isotropic model



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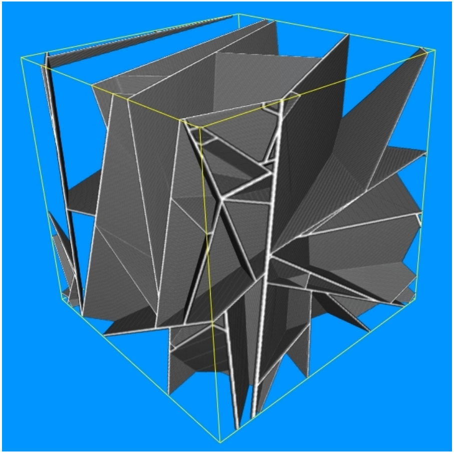

3d isotropic STIT model (Ohser/Redenbach/Sych)

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It has the initial state $Y_0 \wedge W = W$ and the generator

$$\mathbb{L}g(y) = \sum_{z \in y} \int_{[z]} [g(\oslash_{z,h}(y)) - g(y)]\ \Lambda(\mathrm{d}h)$$

for all nonnegative measurable functions g on the set of tessellations of W, and the operator

$$\oslash_{z,h}(y) := (y \setminus \{z\}) \cup \{z \cap h^+, z \cap h^-\}$$

i.e. $\oslash_{z,h}(y)$ is the tessellation that arises from y by splitting the cell z by the hyperplane h.

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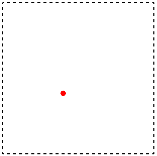
- $Y_{s+t} \stackrel{D}{=} Y_s \boxplus \vec{Y}_t$ for all $s, t > 0$

  ($\boxplus$ ... iteration/nesting of tessellations)

STIT tessellations share several properties with Poisson hyperplane tessellations

Intuitively:
If we are sitting in a fixed point of the space and only see, how the cell around this observation point develops in time,
we cannot distinguish (in distribution)
whether we are sitting in a STIT process or in a Poisson hyperplane tessellation process driven by the same hyperplane measure Λ.



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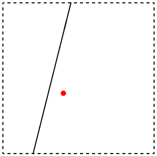

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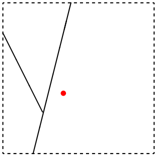

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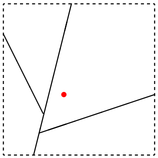

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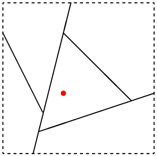

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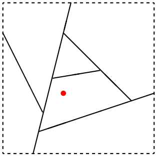

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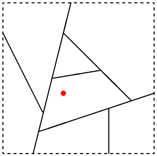

Mecke formula for Poisson hyperplane processes

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Theorem (Mecke formula for Poisson hyperplane processes with birth times)

Let Γ be a Poisson process on $\mathcal{H} \times (0, \infty)$ (of hyperplanes with birth times) with intensity measure $\Lambda(\mathrm{d}h)\,\mathrm{d}s$ and $g : \mathsf{N} \times \mathcal{H} \times (0, \infty) \to \mathbb{R}$ a nonnegative measurable function. Then

$$\int \sum_{(h,s)\in\gamma} g(\gamma, h, s) P_\Gamma(\mathrm{d}\gamma) = \int\int\int g(\gamma + \delta_{(h,s)}, h, s,) P_\Gamma(\mathrm{d}\gamma) \Lambda(\mathrm{d}h)\,\mathrm{d}s.$$

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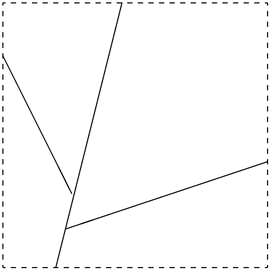
Note: A hyperplane (h, s) of the Poisson hyperplane process (with birth times), does neither depend on the past nor it has an impact on the hyperplanes in future.
In contrast, if a hyperplane divides a cell of STIT at a time s then this has an impact on the cell division after time s.

Mecke type formula for STIT tessellations

Recall the construction of STIT



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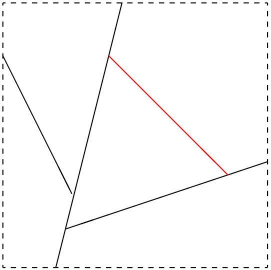

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The birth time of the maximal polytopes is essential!

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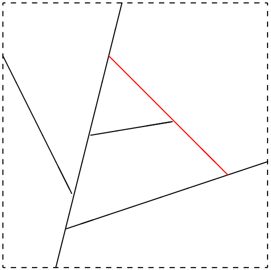

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Theorem (Mecke type theorem for STIT; N./N./Th./W.)

Let M be the process of birth time marked maximal $(d-1)$-polytopes of a STIT tessellation process $(Y_t, t>0)$ driven by a hyperplane measure Λ. Then

$$\int \sum_{(p,s)\in m} g(\qquad p,s)\,\mathbb{P}_M(\mathrm{d}m)$$

$$=\int\int\int \sum_{z\in y_s}$$

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for all nonnegative measurable functions g.

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$$\int \sum_{(p,s)\in m} g\left(z(p,s), p, s\right) \mathbb{P}_M(\mathrm{d}m)$$

$$= \int \int \int \sum_{z\in y_s} g\left(z, z\cap h, s\right) \mathbb{P}_{Y_s}(\mathrm{d}y_s)\, \Lambda(\mathrm{d}h)\, \mathrm{d}s$$

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$$g\big((m^{(1)}_{(+s)} \wedge (z\cap h^+)) \qquad , z, z\cap h, s\big)$$

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$$g\Big(\big(m^{(1)}_{(+s)} \wedge (z \cap h^+)\big) \cup \big(m^{(2)}_{(+s)} \wedge (z \cap h^-)\big), z, z\cap h, s\Big)$$

$$\mathbb{P}_M(\mathrm{d}m^{(1)})\, \mathbb{P}_M(\mathrm{d}m^{(2)})\, \mathbb{P}_{Y_s}(\mathrm{d}y_s)\, \Lambda(\mathrm{d}h)\ \mathrm{d}s$$

for all nonnegative measurable functions g.

The proof uses the 'global construction' (rather involved !!) by Joseph Mecke of STIT tessellations and the Mecke formula for Poisson point processes.

Application: Maximal k-polytopes

STIT process in $\mathbb{R}^d$,

the d-dim. cells are divided by $(d-1)$-dim. hyperplanes

$\Rightarrow$ $(d-1)$-dim. maximal polytopes

the k-dimensional faces of maximal $(d-1)$-polytopes,
$k = 0, ..., d-2$,

maximal k-polytopes

They appear as the intersection of certain sequences of $d-k$ maximal polytopes of dimension $d-1$.

Application

For $k = 0, ..., d-2$ consider a tuple

$$((p_1, s_1), \ldots, (p_{d-k}, s_{d-k}))$$

of maximal $(d-1)$-polytopes together with their birth times with $s_1 < \ldots < s_{d-k}$, and

$\overline{\mathbf{p}} = \bigcap_{i=1}^{d-k} p_i$ is a maximal k-polytope.

Application

For $k = 0, ..., d-2$ consider a tuple

$$((p_1, s_1), \ldots, (p_{d-k}, s_{d-k}))$$

of maximal $(d-1)$-polytopes together with their birth times with $s_1 < \ldots < s_{d-k}$, and

$\overline{\mathbf{p}} = \bigcap_{i=1}^{d-k} p_i$ is a maximal k-polytope.

For a fixed time t and $j = 0, ..., k$ denote by

$$\mathbb{Q}^{(j)}_{(\overline{\mathbf{P}}, \beta_1, \ldots, \beta_{d-k}), t}$$

the distribution of the typical V_j-weighted maximal k-polytope (marked with the birth times) of STIT.

Application

Theorem

Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$ and $t > 0$. The marginal distribution $\mathbb{Q}^{(j)}_{\boldsymbol{\beta},t}$ of the birth times $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{d-k})$ of the typical V_j-weighted maximal k-polytope has the density

$$(s_1, \ldots, s_{d-k}) \mapsto (d-j)(d-k-1)! \frac{s_{d-k}^{k-j}}{t^{d-j}} \mathbf{1}\{0 < s_1 < \ldots < s_{d-k} < t\}$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-k}$.

Corollary

Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, k\}$.
The marginal distribution $\mathbb{Q}^{(j)}_{\beta_{d-k},t}$ of the last birth time of the typical V_j-weighted maximal k-polytope has the density

$$s_{d-k} \mapsto (d-j)\frac{s_{d-k}^{d-j-1}}{t^{d-j}}\mathbf{1}\{0 < s_{d-k} < t\}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

Application

Corollary

For all $s_{d-k} < t$, the conditional distribution

$\mathbb{Q}^{(j)}_{(\beta_1,\ldots,\beta_{d-k-1}),t|\beta_{d-k}=s_{d-k}}$ of the birth times $(\beta_1, \ldots, \beta_{d-k-1})$,

given $\beta_{d-k} = s_{d-k}$ has the density

$$(s_1, \ldots, s_{d-k-1}) \mapsto (d-k-1)!\, s_{d-k}^{-(d-k-1)}\, \mathbf{1}\{0 < s_1 < \ldots < s_{d-k}\}$$

In particular, this conditional distribution does not depend on j, and it is the uniform distribution on the $(d-k-1)$-simplex $\{(s_1, \ldots, s_{d-k-1}) \in \mathbb{R}^{d-k-1} : 0 < s_1 < \ldots < s_{d-k-1} < s_{d-k}\}$.

Theorem

Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$, $t > 0$, $g : \mathcal{P}_k \times (0,t)^{d-k} \to \mathbb{R}$ non-negative and measurable. Then

$$\int g(q,\mathbf{s}) \, \mathbb{Q}^{(j)}_{(\overline{\mathbf{P}},\beta_1,\ldots,\beta_{d-k}),t}(\mathrm{d}(q,\mathbf{s})) = \int \int \int g(q,\mathbf{s})$$

$$\mathbb{Q}^{(j)}_{\beta_{d-k}}(\mathrm{d}s_{d-k})$$

Application

Theorem

Let $d \geq 2$, $k \in \{0, \dots, d-1\}$, $j \in \{0, \dots, k\}$, $t > 0$,
$g : \mathcal{P}_k \times (0,t)^{d-k} \to \mathbb{R}$ non-negative and measurable. Then

$$\int g(q, \mathbf{s}) \, \mathbb{Q}^{(j)}_{(\overline{\mathbf{P}}, \beta_1, \dots, \beta_{d-k}), t}(\mathrm{d}(q, \mathbf{s})) = \int \int \int g(q, \mathbf{s})$$

$$\mathbb{Q}^{(j)}_{\overline{\mathbf{P}}, t | \beta_{d-k} = s_{d-k}}(\mathrm{d}q) \; \mathbb{Q}^{(j)}_{(\beta_1, \dots, \beta_{d-k-1}), t | \beta_{d-k} = s_{d-k}}(\mathrm{d}(s_1, \dots, s_{d-k-1}))$$

$$\mathbb{Q}^{(j)}_{\beta_{d-k}}(\mathrm{d}s_{d-k})$$

Application

Theorem

Let $d \geq 2$, $k \in \{0, \ldots, d-1\}$, $j \in \{0, \ldots, k\}$, $t > 0$, $g : \mathcal{P}_k \times (0,t)^{d-k} \to \mathbb{R}$ non-negative and measurable. Then

$$\int g(q, \mathbf{s})\, \mathbb{Q}^{(j)}_{(\overline{\mathbf{P}}, \beta_1, \ldots, \beta_{d-k}), t}(\mathrm{d}(q, \mathbf{s})) = \int \int \int g(q, \mathbf{s})$$

$$\mathbb{Q}^{(j)}_{\overline{\mathbf{P}}, t | \beta_{d-k} = s_{d-k}}(\mathrm{d}q)\; \mathbb{Q}^{(j)}_{(\beta_1, \ldots, \beta_{d-k-1}), t | \beta_{d-k} = s_{d-k}}(\mathrm{d}(s_1, \ldots, s_{d-k-1}))\; \mathbb{Q}^{(j)}_{\beta_{d-k}}(\mathrm{d}s_{d-k})$$

i.e. the typical V_j-weighted maximal k-polytope $\overline{\mathbf{P}}$ and $(\beta_1, \ldots, \beta_{d-k-1})$ are conditionally independent, given the last birth time $\beta_{d-k} = s_{d-k}$. This can also be interpreted as a Markov property for functionals of the STIT tessellation processes.

Application

Theorem (N./Nguyen/Thäle/Weiß)

Let $d \geq 2$, $t > 0$. The probabilities $\mathrm{p}_{1,1}(n)$ for exactly n nodes in the relative interior of the length weighted typical maximal segment are given by

$$\mathrm{p}_{1,1}(n)$$

$$=(n+1)(d-1)! \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} \frac{s_{d-1}^2}{t^{d-1}} \frac{(d \cdot t - 2s_{d-1} - s_{d-2} - \ldots - s_1)^n}{(d \cdot t - s_{d-1} - s_{d-2} - \ldots - s_1)^{n+2}} \mathrm{d}s_1 \ldots \mathrm{d}s_{d-1}$$

for $n \in \{0, 1, 2, \ldots\}$.