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# A central limit theorem for Lipschitz–Killing curvatures of Gaussian excursions

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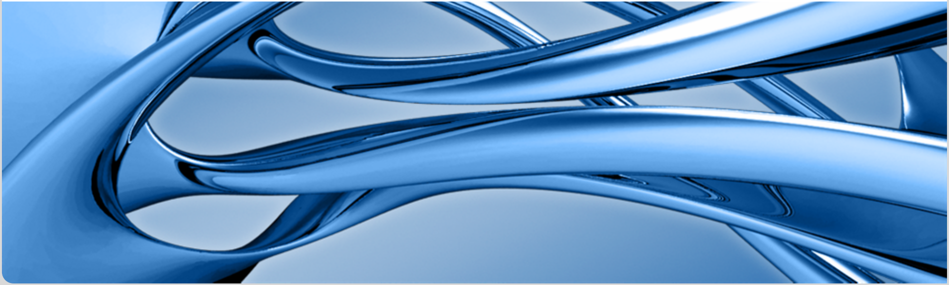

# Fundamentals

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## Definition

A **random field** is a measurable mapping $X\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^{\mathbb{R}^d}, \mathcal{S})$.

If there are functions $m\colon \mathbb{R}^d \to \mathbb{R}$ and $C\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ symmetric and positive-semidefinite, such that

$$(X_{t_1}, \ldots, X_{t_n}) \sim \mathcal{N}_n\left((m(t_i))_{i=1}^n, (C(t_i, t_j))_{i,j=1}^n\right),$$

for $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}^d$, then $X$ is called **Gaussian random field**.

$X$ is called **stationary** if $(X_{t_1}, \ldots, X_{t_n}) \sim (X_{t_1+h}, \ldots, X_{t_n+h})$, $h \in \mathbb{R}^d$.

$X$ is called **isotropic** if $(X_{t_1}, \ldots, X_{t_n}) \sim (X_{\rho t_1}, \ldots, X_{\rho t_n})$, $\rho \in SO(d)$.

$X^{-1}([u, \infty))$ is the **excursion set** of $X$ to the level $u \in \mathbb{R}$.

# Fundamentals

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## Definition [1, Theorem 10.5.6]

For $M \subset \mathbb{R}^d$ nice enough the **Lipschitz–Killing curvatures** $\mathcal{L}_m(M)$ are defined by

$$\mathcal{H}^d(M + B^d_\varepsilon) = \sum_{i=0}^{\dim M} \varepsilon^{d-i} \kappa_{d-i} \mathcal{L}_i(M),$$

where $\kappa_k := \mathcal{H}^k(B^k_1)$, $B^d_r$ is the centered open ball of radius $r > 0$, $\varepsilon < r_c$.

## Example

Take $M$ as the stratified manifold given by

$$\begin{aligned} M &= \mathrm{cl}\, B^d_N \cap X^{-1}([u, \infty)) \\ &= \quad S^{d-1}_N \cap X^{-1}(\{u\}) \quad \cup \quad S^{d-1}_N \cap X^{-1}((u, \infty)) \\ &\quad \cup \quad B^d_N \cap X^{-1}(\{u\}) \quad \cup \quad B^d_N \cap X^{-1}((u, \infty)). \end{aligned}$$

# Conditions for the Main Theorem

(A1) $X$ is a centered, stationary, isotropic Gaussian field with trajectories almost surely of class $\mathcal{C}^3$. The covariance function $C(t) := \mathbb{E}\left[X(t)X(0)\right]$ satisfies $C(0) = 1$ and $-D^2C(0) = I_d$.

(A2) For $0 \neq t \in \mathbb{R}^d$ the covariance matrix of the vector

$$\left(X(t), \left(D^{e_i}X(t)\right)_{i=1}^{d}, \left(D^{e_i e_j}X(t)\right)_{1\leq i\leq j\leq d}, \left(D^{e_i}X(0)\right)_{i=1}^{d}\right)$$

has full rank.

(A3) The mapping
$\psi(t) := \max\left\{\left|D^{e_{j_1}\cdots e_{j_k}}C(t)\right| : k \in \{0,\ldots,4\}, 1 \leq j_1,\ldots,j_k \leq d\right\}$,
$t \in \mathbb{R}^d$, satisfies

$$\psi(t) \overset{\|t\|\to\infty}{\longrightarrow} 0 \text{ and } \psi \in L^1(\mathbb{R}^d).$$

# Main Theorem

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### Theorem

Let $X$ be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (A1)–(A3) and let $m \in \{0, \dots, d-1\}$. Then

$$\frac{\mathcal{L}_m\left(\operatorname{cl} B_N^d \cap X^{-1}([u,\infty))\right) - \mathbb{E}\left[\mathcal{L}_m\left(\operatorname{cl} B_N^d \cap X^{-1}([u,\infty))\right)\right]}{\mathcal{H}^d(B_N^d)^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_m^2)$$

for $N \to \infty$ and some $\sigma_m^2 \geq 0$.

Predecessors:

- Estrade & León ([3]) in the case $m = 0$
- Kratz & León ([5]) in the case $d = 2$ and $m = 1$

# Main Theorem

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## Lemma

Let $X$ be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (A1)–(A3) and let $m \in \{0, \dots, d-1\}$. Then

$$\sigma_m^2 \geq \begin{bmatrix} d \\ d-m \end{bmatrix}^2 (2\pi)^m f(0) H_{d-m}(u)^2 \phi(u)^2.,$$

where

- $f$ is the continuous spectral density of $X$, i.e. $C(t) = \int_{\mathbb{R}^d} e^{i\langle t,\lambda\rangle} f(\lambda)\, d\lambda$,
- $H_k(x) := (-1)^k e^{x^2/2} \frac{\partial}{\partial x} e^{-x^2/2}$ denotes the $k$-th Hermite polynomial,
- $\phi$ denotes the density of a standard normal distribution.

## Strategy of Proof

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Part 1: Establish a Hermite type expansion for the relevant part of the standardized random variable of $\mathcal{L}_m\left(\mathrm{cl}\, B_N^d \cap X^{-1}([u,\infty))\right)$.

Part 2: Embed the Gaussian field $X$ and its derivatives into an isonormal process on a suitable Hilbert space to obtain a representation in terms of stochastic integrals.

Part 3: Verify the conditions of a central limit theorem in Nourdin, Peccati: Normal approximation with Malliavin calculus, [7].

## Strategy of Proof: Part 1

An application of the Crofton formula yields

$$\mathcal{L}_m\left(\mathrm{cl}\, B_N^d \cap X^{-1}([u,\infty))\right) = \int_{A_{d-m}^d} \mathcal{L}_0\left(\mathrm{cl}\, B_N^d \cap X^{-1}([u,\infty)) \cap F\right)\mu(dF),$$

where $\mathcal{L}_0$ is the Euler characteristic. By Morse's theorem (cf. [1, 9.3.5])

$$\begin{aligned}
&\mathcal{L}_0(\mathrm{cl}\, B_N^d \cap X^{-1}([u,\infty)) \cap F) \\
&\quad = \#\{t \in B_N^d \cap F \mid X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X, B_N^d \cap F}(t) \text{ even}\} \\
&\qquad - \#\{t \in B_N^d \cap F \mid X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X, B_N^d \cap F}(t) \text{ odd}\} \\
&\qquad + e(X, N, F),
\end{aligned}$$

where $e(X, N, F)$ is given by

$$\begin{aligned}
&\#\{t \in S_N^{d-1} \cap F \mid X(t) \geq u, \nabla(X|_{S_N^{d-1} \cap F})(t) = 0, \iota_{-X, S_N^{d-1} \cap F}(t) \text{ even}\} \\
&\quad - \#\{t \in S_N^{d-1} \cap F \mid X(t) \geq u, \nabla(X|_{S_N^{d-1} \cap F})(t) = 0, \iota_{-X, S_N^{d-1} \cap F}(t) \text{ odd}\}.
\end{aligned}$$

## Strategy of Proof: Part 1

### Lemma

Let $X$ satisfy the conditions (A1)-(A2). Then

$$\mathcal{H}^d(B_N^d)^{-1} \int_{A_{d-m}^d} e(X, N, F)\, \mu(dF) \xrightarrow{\mathbb{P}} 0 \quad \text{as } N \to \infty.$$

### Proof

Apply the Rice formula (cf. [2, Theorem 6.2]), which states that for a nice enough Gaussian field $Z\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and a measurable set $B \subset \mathbb{R}^d$

$$\mathbb{E}\left[\#\{t \in B \mid Z(t) = 0\}\right] = \int_B \mathbb{E}\left[|\det Z'(t)| \mid Z(t) = 0\right] p_{Z(t)}(0)\, dt.$$

## Strategy of Proof: Part 1

Hence, in the following we consider

$$\zeta_{m,N} := \int_{A^d_{d-m}} \#\{t \in B^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X, B^d_N \cap F}(t) \text{ even}\} - \#\{t \in B^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X, B^d_N \cap F}(t) \text{ odd}\}\, \mu(dF)$$

and approximate this random variable by

$$\zeta^{\varepsilon}_{m,N} := (-1)^{d-m} \int_{A^d_{d-m}} \int_{B^d_N \cap F} \delta_\varepsilon(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \times \det(D^2(X|_F)(t))\, dt\, \mu(dF),$$

where $\delta_\varepsilon \colon \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \frac{1}{\varepsilon^{d-m}\kappa_{d-m}} \mathbb{1}_{B^d_\varepsilon}(x)$, so that for $E \in G^d_{d-m}$ and $f : E \to \mathbb{R}$ continuous $\lim_{\varepsilon \to 0} \int_E \delta_\varepsilon(x) f(x)\, dx = f(0)$.

# Strategy of Proof: Part 1

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Hence, in the following we consider

$$\zeta_{m,N} := \int_{A^d_{d-m}} \#\{t \in B^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X,B^d_N \cap F}(t) \text{ even}\} \\ - \#\{t \in B^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_{-X,B^d_N \cap F}(t) \text{ odd}\}\, \mu(dF)$$

and approximate this random variable by

$$\zeta^{\varepsilon}_{m,N} := (-1)^{d-m} \int_{A^d_{d-m}} \int_{B^d_N \cap F} \delta_{\varepsilon}(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \\ \times \det(D^2(X|_F)(t))\, dt\, \mu(dF).$$

## Lemma

Let $X$ satisfy (A1) and (A2). Then

$$\zeta^{\varepsilon}_{m,N} \xrightarrow{L^2(\mathbb{P})} \zeta_{m,N} \quad \text{as } \varepsilon \to 0.$$

Proof: Applications of Rice formulas and lengthy calculations.

## Strategy of Proof: Part 1

Lemma

Let $\varepsilon > 0$. Then

$$\zeta^{\varepsilon}_{m,N} = \int_{G^d_{d-m}} \int_{B^d_N} G_\varepsilon(Y(F,t))\, dt\, \nu(dF),$$

for a suitable $G_\varepsilon \in L^2\left(\mathbb{R}^D, \phi_D(x)dx\right)$, where

$$Y(F,t) := \Lambda^{-1}\left(\left(D^{v_i}X(t)\right)_{i=1}^{d-m}, \left(D^{v_i v_j}X(t)\right)_{1\le i\le j\le d-m}, X(t)\right)$$

for $t \in \mathbb{R}^d$, $F \in G(d, d-m)$ and $(v_i)$ denoting an orthonormal basis of $F$, where $\Lambda$ is the root of the covariance matrix of the vector $\left(\left(D^{v_i}X(t)\right)_{i=1}^{d-m}, \left(D^{v_i v_j}X(t)\right)_{1\le i\le j\le d-m}, X(t)\right)$.

## Strategy of Proof: Part 1

The Hermite expansion

$$G_\varepsilon = \sum_{q=0}^{\infty} \sum_{n \in \mathbb{N}^D, |n|=q} c(G_\varepsilon, n) \widetilde{H}_n,$$

with $\widetilde{H}_n(x_1, \ldots, x_D) = \prod_{i=1}^{D} H_{n_i}(x_i)$, $x \in \mathbb{R}^D$, yields

### Lemma

Let $X$ satisfy (A1) and (A2) and let $\varepsilon > 0$. Then

$$\zeta_{m,N}^{\varepsilon} \overset{L^2(\mathbb{P})}{=} \sum_{q \geq 0} \sum_{n \in \mathbb{N}^D, |n|=q} \int_{G_{d-m}^d} c(G_\varepsilon, n) \int_{B_N^d} \widetilde{H}_n(Y(F,t))\, dt\, \nu(dF).$$

### Theorem

Let $X$ satisfy (A1) and (A2) and $c(n) := \lim_{\varepsilon \to 0} c(G_\varepsilon, n)$. Then

$$\zeta_{m,N} \overset{L^2(\mathbb{P})}{=} \sum_{q \geq 0} \sum_{n \in \mathbb{N}^D, |n|=q} \int_{G_{d-m}^d} c(n) \int_{B_N^d} \widetilde{H}_n(Y(F,t))\, dt\, \nu(dF).$$

## Strategy of Proof: Part 2

### Definition

We define the real Hilbert space

$$\mathfrak{H} := \left\{ h\colon \mathbb{R}^d \to \mathbb{C} \mid h(-x) = \overline{h(x)},\ \int_{\mathbb{R}^d} |h(x)|^2 f(x)\, dx < \infty \right\}$$

with the inner product $\langle g, h \rangle_{L^2(f(x)dx)} := \int_{\mathbb{R}^d} g(x)\overline{h(x)} f(x)\, dx$ and the isonormal process $W$, i.e. the centered Gaussian field on $\mathfrak{H}$ with

$$\mathbb{E}\left[W(f)W(g)\right] = \langle f, g \rangle_{L^2(f(x)dx)}.$$

### Lemma

There are explicitly known functions $\varphi^F_{t,j} \in \mathfrak{H}$, so that

$$\mathbb{E}[Y_i(F,t)Y_j(F',t')] = \langle \varphi^F_{t,i}, \varphi^{F'}_{t',j} \rangle_{L^2(f(x)dx)}$$

and therefore

$$Y(\cdot, \cdot\cdot) \overset{\mathcal{D}}{=} \left(W(\varphi^{\cdot}_{\cdot\cdot,1}), \ldots, W(\varphi^{\cdot}_{\cdot\cdot,D})\right).$$

# Strategy of Proof: Part 2

This leads to

$$\prod_{i=1}^{D} H_{n_i}(Y_i(\cdot,\cdot\cdot)) \overset{\mathcal{D}}{=} \prod_{i=1}^{D} H_{n_i}(W(\varphi^{\cdot}_{\cdot\cdot,i})) = I_q(\varphi^{\cdot\otimes n_1}_{\cdot\cdot,1} \otimes \ldots \otimes \varphi^{\cdot\otimes n_D}_{\cdot\cdot,D}),$$

where $I_q$ denotes the $q$-th multiple Wiener–Itô integral. Hence, we obtain the representation

$$\frac{\zeta_{m,N} - \mathbb{E}[\zeta_{m,N}]}{\mathcal{H}^d(B_N^d)^{1/2}} \overset{\mathcal{D}}{=} \sum_{q=1}^{\infty} I_q(g_{N,q}),$$

where

$$g_{N,q} := \frac{1}{\mathcal{H}^d(B_N^d)^{1/2}} \sum_{k\in\{1,\ldots,D\}^q} \int_{G^d_{d-m}} b(k) \int_{B_N^d} \varphi^F_{t,k_1} \otimes \ldots \otimes \varphi^F_{t,k_q}\, dt\, \nu(dF).$$

## Strategy of Proof: Part 3

Theorem 6.3.1 in [7]

Let $F_N \in L^2(\mathbb{P})$, for $N \in \mathbb{N}$, such that $\mathbb{E}[F_N] = 0$. Then there exist functions $g_{N,q} \in \mathfrak{H}^{\odot q}$, for $N, q \in \mathbb{N}$, such that $F_N = \sum_{q \geq 1} I_q(g_{N,q})$. Suppose that the following conditions

(i) for fixed $q \geq 1$ there exists $\sigma_q^2 \geq 0$ such that $q! \|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}} \xrightarrow{N \to \infty} \sigma_q^2$,

(ii) $\sigma^2 := \sum_{q \geq 1} \sigma_q^2 < \infty$,

(iii) for all $q \geq 2$ and $r = 1, \ldots, q-1$: $\|g_{N,q} \otimes_r g_{N,q}\|_{\mathfrak{H}^{\otimes(2q-2r)}} \xrightarrow{N \to \infty} 0$,

(iv) $\lim_{Q \to \infty} \limsup_{N \to \infty} \sum_{q=Q+1}^{\infty} q! \|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}} = 0$

are true. Then $F_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$.

Verification of (i)-(iv): Heavily relies on (A3) and uses the ideas of Estrade & León with extra attention for the integration on $G^d_{d-m}$ and the different observation window $B^d_N$.

## Ongoing work

- A multivariate central limit theorem, i.e.

$$\frac{(\zeta_{0,N},\dots,\zeta_{d-1,N}) - \mathbb{E}\left[(\zeta_{0,N},\dots,\zeta_{d-1,N})\right]}{\mathcal{H}^d(B_N^d)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}_d(0,\Sigma)$$

  and conditions that guarantee the positive definiteness of $\Sigma$.
- A multivariate central limit theorem for integrated level functionals, i.e. establish for

$$\Psi^h(X,A) := \int_{\mathbb{R}} \int_{X^{-1}(u)\cap A} h(\nabla X(t), D^2X(t), X(t))\, \mathcal{H}^{d-1}(dt)\, du$$

  where $h\colon \mathbb{R}^D \to \mathbb{R}^k$ nice enough, the limit

$$\frac{\Psi^h(X,A_N) - \mathbb{E}\left[\Psi^h(X,A_N)\right]}{\mathcal{H}^d(A_N)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}_k(0,\Sigma^h) \quad \text{as } A_N \nearrow \mathbb{R}^d$$

  and conditions that guarantee the positive definiteness of $\Sigma^h$.

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