Günter Last
Institut für Stochastik
Karlsruher Institut für Technologie

DFG FORSCHERGRUPPE
Geometry & Physics of
Spatial Random Systems

Independent edge marking of Poisson processes

Günter Last (Karlsruhe)

joint work with

Franz Nestmann (Karlsruhe) and Matthias Schulte (Bern)

presented at the

19th Workshop on Stochastic Geometry, Stereology and Image Analysis

Marseille, May 15–19, 2017

1. Fock space representation

Setting

η is a Poisson process on some measurable space $(\mathbb{X}, \mathcal{X})$ with σ-finite intensity measure λ. This is a random element in the space **N** of all integer-valued σ-finite measures on $\mathbb{X}$, equipped with the usual σ-field (and distribution Π_λ) with the following two properties

- The random variables $\eta(B_1), \ldots, \eta(B_m)$ are stochastically independent whenever $B_1, \ldots, B_m$ are measurable and pairwise disjoint.
-
$$\mathbb{P}(\eta(B) = k) = \frac{\lambda(B)^k}{k!} \exp[-\lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{X},$$

  where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

Definition (Difference operator)

For a measurable function $f : \mathbf{N} \to \mathbb{R}$ and $x \in \mathbb{X}$ we define a function $D_x f : \mathbf{N} \to \mathbb{R}$ by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu).$$

For $x_1, \dots, x_n \in \mathbb{X}$ we define $D^n_{x_1,\dots,x_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f,$$

where $D^1 := D$ and $D^0 f = f$.

Lemma

For any $f \in L^2(\mathbb{P}_\eta)$

$$T_n f(x_1, \dots, x_n) := \mathbb{E} D^n_{x_1,\dots,x_n} f(\eta),$$

defines a function $T_n f \in L^2_s(\lambda^n)$.

Theorem (L. and Penrose '10)

Let $f, g \in L^2(\mathbb{P}_\eta)$. Then

$$\mathbb{E}f(\eta)g(\eta) = (\mathbb{E}f(\eta))(\mathbb{E}g(\eta)) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (T_n f)(T_n g)\, d\lambda^n.$$

2. A covariance identity

Definition

Assume that the Poisson process η is proper, that is of the form

$$\eta = \sum_{n=1}^{\eta(\mathbb{X})} \delta_{X_n}$$

for some random elements $X_1, X_2, \dots$. Let $U_1, U_2, \dots$ be independent random variables, uniformly distributed on $[0, 1]$ and independent of $(\eta(\mathbb{X}), (X_n)_{n \geq 1})$. For each $t \in [0, 1]$ define a t-thinning of η by

$$\eta_t := \sum_{n=1}^{\eta(\mathbb{X})} \mathbf{1}\{U_n \leq t\} \delta_{X_n}.$$

Definition

Let L^2_η denote the space of all random variables $F \in L^2(\mathbb{P})$ such that $F = f(\eta)$ $\mathbb{P}$-almost surely, for some measurable function (representative) $f : \mathbf{N} \to \mathbb{R}$.

Definition

Let $F \in L^2_\eta$ have representative f. Define $D_x F := D_x f(\eta)$ for $x \in \mathbb{X}$ The mapping $(\omega, x) \mapsto D_x F(\omega)$ is denoted by DF.

Theorem

Let $F \in L^2_\eta$ and $G \in L^2_\eta$ be such that $DF, DG \in L^2(\mathbb{P} \otimes \lambda)$. Then

$$\mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G] = \mathbb{E}\Big[\iint_0^1 (D_x F)(P_t D_x G)\, dt\, \lambda(dx)\Big],$$

where

$$P_t F := \mathbb{E}\Big[\int f(\eta_t + \mu)\, \Pi_{(1-t)\lambda}(d\mu) \,\Big|\, \eta\Big].$$

Theorem

Let $F \in L^2_\eta$ and $G \in L^2_\eta$. Then

$$\mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G] = \mathbb{E}\Big[\iint_0^1 \mathbb{E}[D_x F \mid \eta_t]\mathbb{E}[D_x G \mid \eta_t]\, dt\, \lambda(dx)\Big].$$

3. The Poincaré inequality

Theorem (Chen '85; Wu '00; L. and Penrose '11)

Let $F \in L^2_\eta$. *Then*

$$\mathbb{V}\mathrm{ar}[F] \le \mathbb{E} \int (D_x F)^2 \lambda(dx).$$

Equality holds iff F *is a linear function of* η.

4. Chaos expansion of Poisson functionals

Definition

Let $n \in \mathbb{N}$ and $g \in L^2(\lambda^n)$. Then $I_n(g)$ denotes the multiple Wiener-Itô integral of g w.r.t. the compensated Poisson process $\eta - \lambda$. For $c \in \mathbb{R}$ let $I_0(c) := c$. These integrals have the properties

$$\mathbb{E} I_n(g) I_n(h) = n! \langle \tilde{g}, \tilde{h} \rangle_n, \quad n \in \mathbb{N}_0,$$
$$\mathbb{E} I_m(g) I_n(h) = 0, \quad m \neq n.$$

Here

$$\tilde{g}(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\pi \in \Sigma_n} g(x_{\pi(1)}, \dots, x_{\pi(n)})$$

denotes the symmetrization of g.

Remark (Surgailis '84)

Let $n \geq 1$ and $g \in L^1(\lambda^n) \cap L^2(\lambda^n)$. Then

$$I_n(g) = \sum_{J \subset [n]} (-1)^{n-|J|} \int g(x_1, \dots, x_n) \eta^{(|J|)}(dx_J) \lambda^{n-|J|}(dx_{J^c}),$$

where $[n] := \{1, \dots, n\}$, $J^c := [n] \setminus J$ and $x_J := (x_j)_{j \in J}$ and $\eta^{(m)}$ is the m-th factorial moment measure associated with η:

$$\eta^{(m)}(B) := \int \cdots \int \mathbf{1}_B(x_1, \dots, x_m) \Big(\eta - \sum_{j=1}^{m-1} \delta_{x_j}\Big)(dx_m)$$

$$\dots (\eta - \delta_{x_1})(dx_2) \eta(dx_1).$$

Theorem (Wiener '38; Itô '56; Y. Ito '88; L. and Penrose '11)

For any $F \in L^2_\eta$ there are uniquely determined $f_n \in L^2_s(\lambda^n)$ such that $\mathbb{P}$-a.s.

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n).$$

Moreover, we have $f_n = \frac{1}{n!} T_n f$, where f is a representative of F.

5. Normal approximation of Poisson functionals

Definition

Let X be a random variable and N standard normal. The Wasserstein distance between the laws of X and N is defined by

$$d_W(X,N) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(X) - \mathbb{E}h(N)|.$$

The Kolmogorov distance is given by

$$d_K(X,N) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \le x) - \mathbb{P}(N \le x)|.$$

Theorem (Peccati, Solé, Taqqu, Utzet '10; L., Peccati, Schulte '16)

Suppose that $F \in L^2_\eta$ *satisfies* $DF \in L^2(\mathbb{P} \otimes \lambda)$ *and* $\mathbb{E}[F] = 0$. *Then*

$$d_W(F, N) \le \mathbb{E}\left[\left|1 - \iint_0^1 (P_t D_x F)(D_x F)\, dt\, \lambda(dx)\right|\right] + \mathbb{E}\left[\iint_0^1 |P_t D_x F|(D_x F)^2\, dt\, \lambda(dx)\right].$$

6. Edge markings of Poisson processes

Setting

- η is a Poisson process on a Borel space $\mathbb{X}$ with a σ-finite and diffuse intensity measure λ.
- There is a measurable partial ordering $<$ on $\mathbb{X}$, that orders the points of $\mathbb{X}$ λ-a.e.
- Let $\mathbb{X}^{[2]}$ denote the space of all sets $e \subset \mathbb{X}$ containing exactly two elements.
- Let $(\mathbb{M}, \mathcal{M})$ be a further Borel space equipped with a probability measure $\mathbb{Q}$.

Definition

Let $Z_{m,n}$, $m, n \in \mathbb{N}$, be independent random elements of $\mathbb{M}$ with common distribution $\mathbb{Q}$. Then

$$\xi := \sum_{m,n=1}^{\eta(\mathbb{X})} \mathbf{1}\{X_m < X_n\}\delta_{(\{X_m,X_n\},Z_{m,n})},$$

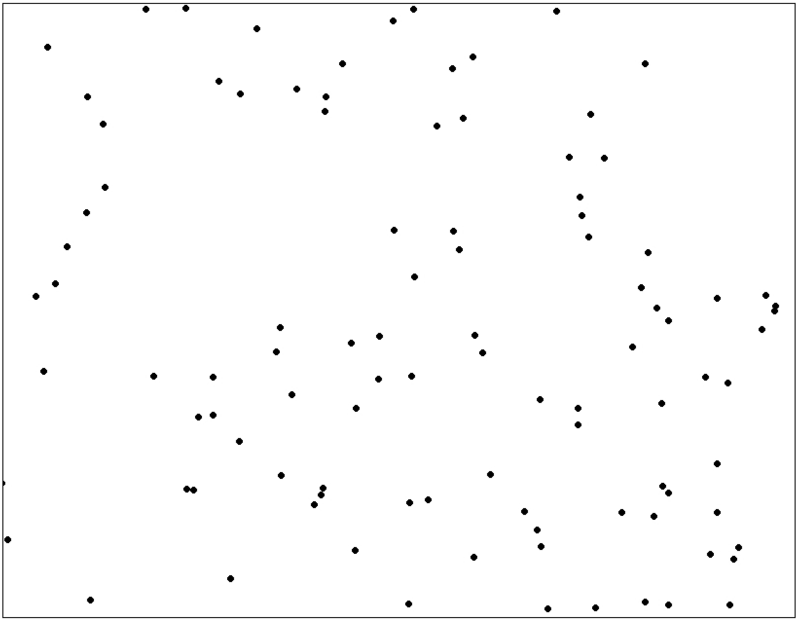
(a point process on $\mathbb{X}^{[2]} \times \mathbb{M}$), is called an independent edge marking of η.

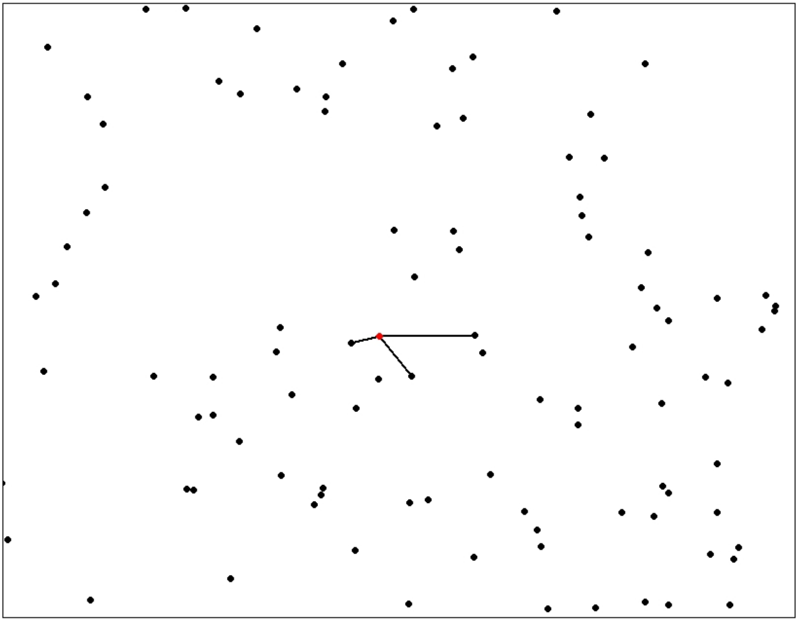
7. The random connection model

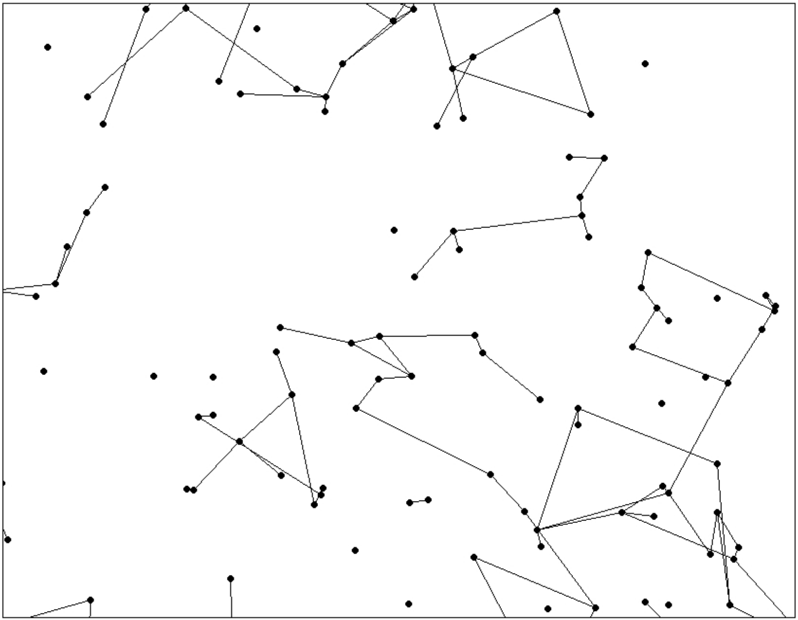
Definition

Let $\varphi : \mathbb{X} \times \mathbb{X} \to [0, 1]$ be a measurable and symmetric connection function. Given η, connect any two distinct points $x, y \in \eta$ with probability $\varphi(x, y)$ independently of all other pairs. This gives the random connection model $\Gamma_\varphi(\eta) := (\eta, \chi)$, where χ is the point process of edges. More formally we can choose $\mathbb{M} = [0, 1]$ and set

$$\chi := \{\{X_m, X_n\} : X_m < X_n, Z_{m,n} \leq \varphi(X_m, X_n)\}.$$







Definition

Let $Z_{m,n}$, $m, n \in \mathbb{Z}$, be independent random elements of $\mathbb{M}$ with common distribution $\mathbb{Q}$. For $k \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{X}$ and $I \subset [k]$ define a point process $\xi_{(x_i)_{i \in I}}$ on $\mathbb{X}^{[2]} \times \mathbb{M}$ by

$$\xi_{(x_i)_{i \in I}} := \sum_{m,n \in [\eta(\mathbb{X})] \cup \{-i : i \in I\}} \mathbf{1}\{X_m < X_n\} \delta_{(\{X_m, X_n\}, Z_{m,n})},$$

where $X_{-i} := x_i$, $i \in I$, and $[m] := \{k \in \mathbb{N} : k \leq m\}$ for $m \in \mathbb{N} \cup \{\infty\}$.

Definition

Let L_ξ denote the space of all $\sigma(\xi)$-measurable random variables. For each $F \in L_\xi$ there is a measurable $f\colon \mathbf{N}(\mathbb{X}^{[2]} \times \mathbb{M}) \to \mathbb{R}$ such that $F = f(\xi)$ a.s. We call f a representative of F.

Definition

Let $F \in L_\xi$ have representative f. For each $k \in \mathbb{N}$ and all $x_1, \dots, x_k \in \mathbb{X}$ we define a random variable $\Delta^k_{x_1,\dots,x_k} F$ by

$$\Delta^k_{x_1,\dots,x_k} F := \sum_{I \subset [k]} (-1)^{k-|I|} f(\xi_{(x_i)_{i \in I}}).$$

Theorem (Fock space inequality, L., Nestmann and Schulte '17+)

Let $F \in L_\xi$ be such that $\mathbb{E}F^2 < \infty$. Then

$$\mathbb{V}\text{ar}\, F \geq \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{E}\Big[\mathbb{E}\big[\Delta^n_{x_1,\dots,x_n} F \mid \mathcal{F}_{x_1,\dots,x_n}\big]^2\Big] \lambda^n(d(x_1,\dots,x_n)),$$

where $\mathcal{F}_{x_1,\dots,x_n}$ is the σ-field generated by the marks on the complete graph supported by $\{x_1,\dots,x_n\}$.

Theorem (Poincaré inequality, L., Nestmann and Schulte '17+)

Let $F \in L_\xi$ *satisfy* $\mathbb{E}F^2 < \infty$. *Then*

$$\mathbb{V}\mathrm{ar}\, F \le \mathbb{E} \int (\Delta_x F)^2 \, \lambda(dx).$$

Theorem (L., Nestmann and Schulte '17+)

Let $F \in L_\xi$ be such that $\mathbb{E}F^4 < \infty$, $\mathbb{E}F = 0$ and $\mathbb{V}\mathrm{ar}\, F = 1$ and let N be standard normal. Then, under further integrability assumptions on F,

$$d_W(F, N) \le \gamma_1 + \gamma_2 + \gamma_3,$$

where

$$\gamma_1^2 := 4 \int \left[\mathbb{E}(\Delta_{x_1} F)^2 (\Delta_{x_2} F)^2\right]^{1/2} \left[\mathbb{E}(\Delta^2_{x_1,x_3} F)^2 (\Delta^2_{x_2,x_3} F)^2\right]^{1/2} d\lambda^3,$$

$$\gamma_2^2 := \int \mathbb{E}(\Delta^2_{x_1,x_3} F)^2 (\Delta^2_{x_2,x_3} F)^2 \, d\lambda^3,$$

$$\gamma_3 := \int \mathbb{E}|\Delta_x F|^3 \, \lambda(dx).$$

Idea of the proof:

1. Represent ξ as a function of a suitable independent marking η^* of the Poisson process η.
2. Write $F^* = T(\eta^*)$ such that $F \stackrel{d}{=} F^*$.
3. Apply the second order Poincaré inequality from L., Peccati, Schulte (2016) to F^*.
4. Establish a connection between $(\Delta F, \Delta^2 F)$ and $(DF^*, D^2 F^*)$. (There is no commutativity!)

10. References

- Chatterjee, S. (2009). Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143**, 1-40.
- Chen, L. (1985). Poincaré-type inequalities via stochastic integrals. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **69**, 251-277.
- Itô, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments. *Trans. Amer. Math. Soc.* **81**, 253-263.
- Ito, Y. (1988). Generalized Poisson functionals. *Probab. Th. Rel. Fields* **77**, 1-28.

- Last, G., Nestmann, F. and Schulte, M. (2017+). Second order properties and normal approximation of functions of edge-marked Poisson processes with an application to the random connection model. In preparation.
- Last, G., Peccati, G. and Schulte, M. (2016). Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. *Probab. Th. Rel. Fields* **165**, 667-723.
- Last, G. and Penrose, M.D. (2011). Fock space representation, chaos expansion and covariance inequalities for general Poisson processes. *Probab. Th. Rel. Fields* **150**, 663-690.
- Last, G. and Penrose, M.D. (2017). *Lectures on the Poisson Process.* Cambridge University Press, to appear.

- Meester, R. and Roy, R. (1996). *Continuum Percolation.* Cambridge University Press, Cambridge.
- Peccati, G., Solé, J.L., Taqqu, M.S. and Utzet, F. (2010). Stein's method and normal approximation of Poisson functionals. *Ann. Probab.* **38**, 443-478.
- Wu, L. (2000). A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab. Theory Related Fields* **118**, 427-438.

Announcement of the Conference

Geometry and Physics of Spatial Random Systems

September 10–15, 2017
Bad Herrenalb (Germany)