Stable Transports between Stationary Random Measures

Ali Khezeli
Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Based on Ph.D. thesis at Sharif University of Technology under supervision of Mir-Omid Haji-Mirsadeghi

Motivation

Does there exist a transport kernel for any pair of measures φ and ψ on $\mathbb{R}^d$ with $\varphi(\mathbb{R}^d) = \psi(\mathbb{R}^d)$ satisfying the following?

(1) Depends on φ and ψ in a measurable and translation-invariant way,

(2) and balances between φ and ψ:

$$T(x,B), \quad \forall x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d),$$
$$T(x,\mathbb{R}^d) = 1,$$
$$\int T(x,B)\varphi(dx) = \psi(B).$$

Trivial for finite measures, but false for general measures!

Almost Sure Existence for Random Measures

Theorem 1 [2]

If Φ and Ψ are ergodic stationary random measures on $\mathbb{R}^d$, then there exist a transport kernel satisfying (1) and almost surely (2) for Φ and Ψ, if and only if the intensities λ_Φ and λ_Ψ are equal.

This theorem in [2] is merely based on [H. Thorisson, 1996].

Main Problem

Problem

Construct a transport kernel as in Theorem 1 explicitly.

Many solutions in the last decade for the case of the Lebesgue measure and a point process. A break-through is [1].

Our Method

Idea

Impose the condition

$$T(x,\cdot) \le \psi(\cdot)$$

and relax the equalities of (2).

Work with the Radon-Nikodym derivative f:

$$T(x,B) = \int_B f(x,\xi)\psi(d\xi).$$

Definition

Relaxation of (2): Sub-balancing:

$$\int f(x,\xi)\psi(d\xi) \le 1, \ \forall x \in \mathbb{R}^d$$
$$\int f(x,\xi)\varphi(dx) \le 1, \ \forall \xi \in \mathbb{R}^d$$

Constrained (sub-balancing transport) density: additionally,

$$0 \le f \le 1$$

Notation:

- site: a point in the support of φ.
- center: a point in the support of ψ.

Algorithm overview to construct a constrained density: Infinitely many stages, at each stage:

- Each site *applies* to the *as close as possible* centers, taking into account the previous rejections.
- Each center *rejects* some portion of applications if it has reached its capacity, keeping the *as close as possible* applications.

Algorithm (Site-Optimal Transport Density)

Given measures φ and ψ on $\mathbb{R}^d$, define

$$f_s(x,\xi) := \lim_{n\to\infty} A_n(x,\xi) - R_n(x,\xi),$$

where the functions are defined recursively: Let $R_0(x,\xi) := 0$ and for each $n \ge 1$,

(i) Define the *application radius* of each site x_0 by

$$a_n(x_0) := \sup\left\{a : \int_{B_a(x_0)} (1 - R_{n-1}(x_0,\xi))\,\psi(d\xi) \le 1\right\},$$

and the *application function* by

$$A_n(x_0,\xi) := \begin{cases} 1 & |x_0-\xi| < a_n(x_0), \\ cR_{n-1}(x_0,\xi) + (1-c) & |x_0-\xi| = a_n(x_0), \\ 0 & |x_0-\xi| > a_n(x_0), \end{cases}$$

where $c = c_n(x_0)$ is the constant in $[0,1]$ s.th.

$$\int_{\mathbb{R}^d} (A_n(x_0,\xi) - R_{n-1}(x_0,\xi))\,\psi(d\xi) = 1 \quad \text{if } a_n(x_0) < \infty.$$

(ii) Define the *rejection radius* of each center ξ_0 by

$$r_n(\xi_0) := \sup\left\{r : \int_{B_r(\xi_0)} A_n(x,\xi_0)\varphi(dx) \le 1\right\},$$

and its *rejection function* by

$$R_n(x,\xi_0) := \begin{cases} 0 & |x-\xi_0| < r_n(\xi_0), \\ c'A_n(x,\xi_0) & |x-\xi_0| = r_n(\xi_0), \\ A_n(x,\xi_0) & |x-\xi_0| > r_n(\xi_0), \end{cases}$$

where $c' = c'_n(\xi_0)$ is the constant in $[0,1]$ s.th.

$$\int_{\mathbb{R}^d} (A_n(x,\xi_0) - R_n(x,\xi_0))\,\varphi(dx) = 1 \quad \text{if } r_n(\xi_0) < \infty.$$

Definition

A constrained density f is called **stable** if there is no pair (x_0,ξ_0) of a site and a center that both *desire* each other, where site x_0 **desires** center ξ_0 when $f(x_0,\xi_0) < 1$ and either

- **unsatisfied**: $\int_{\mathbb{R}^d} f(x_0,\xi)\psi(d\xi) < 1$
- or $\exists \xi_1 \in \mathbb{R}^d : |x_0 - \xi_1| > |x_0 - \xi_0|$ and $f(x_0,\xi_1) > 0$.

center ξ_0 desires site x_0 with a similar condition.

Theorem

1. The site-optimal density is stable.
2. In Theorem 1, if a stable constrained density satisfies (1) (e.g. the site-optimal density) and $\lambda_\Phi = \lambda_\Psi$, then it almost surely balances between Φ and Ψ.

Main Tool. The mass transport principle:

$$\lambda_\Phi \mathbb{E}_\Phi\left[\Psi_0(F,t)\right] = \lambda_\Psi \mathbb{E}_\Psi\left[\Phi^0(F,t)\right],$$

where $\mathbb{E}_\Phi$ is expectation w.r.t. the Palm distribution of Φ and

$$\psi_x(f,t) := \int_{\mathbb{R}^d} f(x,\xi)1_{|x-\xi|\le t}\psi(d\xi).$$

Proof sketch.

- Ergodicity $\Rightarrow$ $\mathbb{E}_\Phi\left[\Psi_0(F,\infty)\right] < 1$ (resp. $= 1$) iff unsatisfied sites have infinite (resp. zero) Φ-measure.
- Ergodicity $\Rightarrow$ $\mathbb{E}_\Psi\left[\Phi^0(F,\infty)\right] < 1$ (resp. $= 1$) iff unsatisfied centers have infinite (resp. zero) Ψ-measure.
- Both cannot happen due to stability.
- By the mass transport principle for $t = \infty$, Equality happens in both.

Properties of Stable Constrained Densities

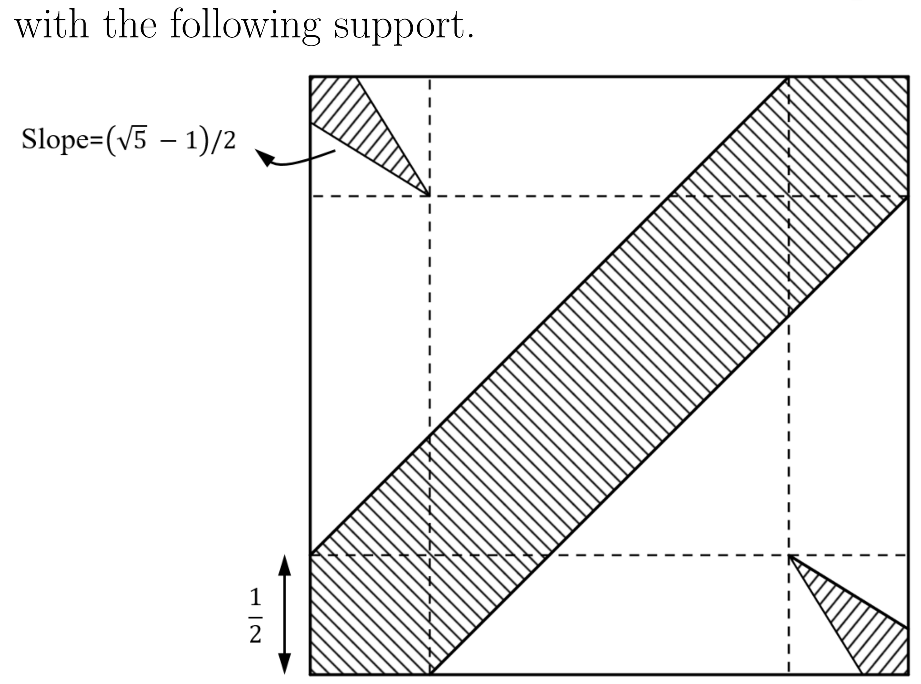
- Monotonicity w.r.t. φ and ψ.
- Optimality: the site-optimal density is the best for sites and worst for centers: For any stable constrained density f, for a.e. site and center, for all $t \in [0,\infty]$,

$$\psi_x(f_s,t) \ge \psi_x(f,t)$$
$$\varphi^\xi(f,t) \ge \varphi^\xi(f_s,t).$$

- In the stationary case, stable constrained density for Φ and Ψ is a.s. unique.
- In the stationary stable case, a.s. the **territories** (i.e. supports of $f(x,\cdot)$ and $f(\cdot,\xi)$) are bounded for a.e. x and ξ.

Examples

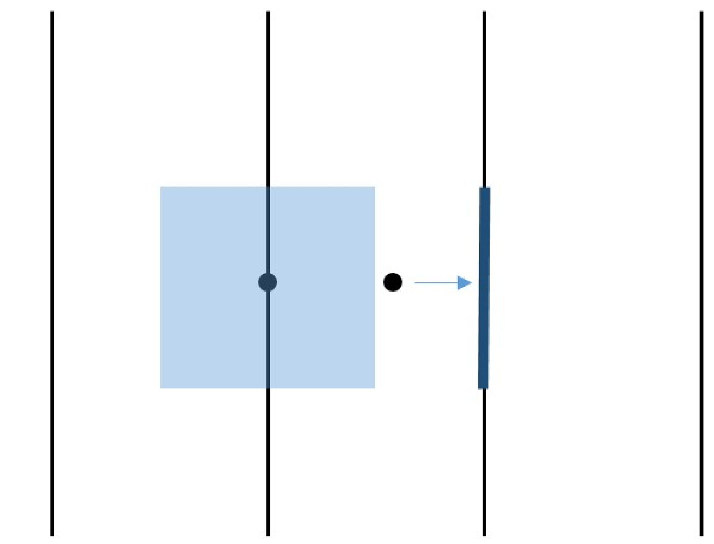
Example 1. Uniform on interval $[0,\alpha]$ for $\alpha \ge \frac{3}{2}$. f_s is $\{0,1\}$-valued with the following support.



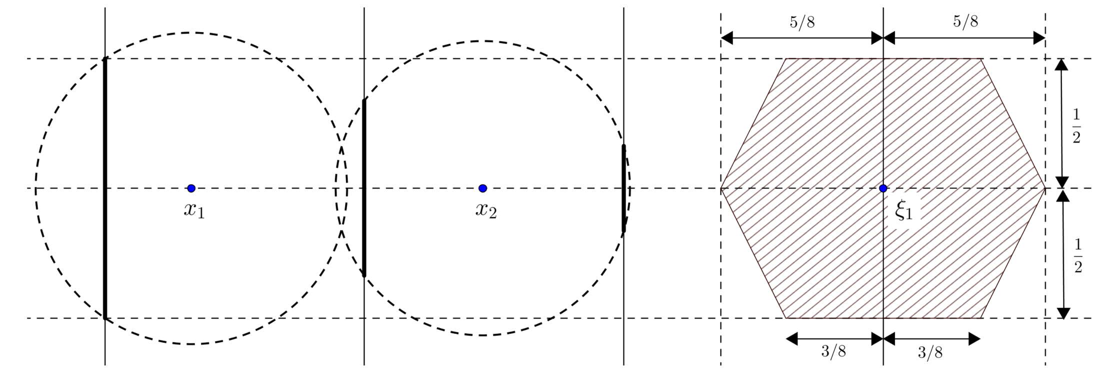
Example 2. Let Φ and Ψ be ergodic point processes, regarded as counting measures. Let $\Phi' := \frac{1}{\lambda_\Phi}\Phi, \quad \Psi' := \frac{1}{\lambda_\Psi}\Psi$,
If $\frac{\lambda_\Phi}{\lambda_\Psi} \notin \mathbb{Z}$, then there is no balancing allocation (i.e. $T(x,\cdot) = \delta_{\tau(x)}$) for Φ' and Ψ' satisfying Theorem 1.

In Examples 3 and 4, f_s is $\{0,1\}$-valued. The images show some territories.

Example 3.
Φ: The Lebesgue measure,
Ψ: 1-dim Hausdorff measure on $\mathbb{Z} \times \mathbb{R}$.
A non-stable constrained density:



The site-optimal density is obtained in one step ($f_s = A_1$):



Example 4.
Φ: The Lebesgue measure,
Ψ: an ergodic point process.
The site-optimal algorithm is a generalization of [1].

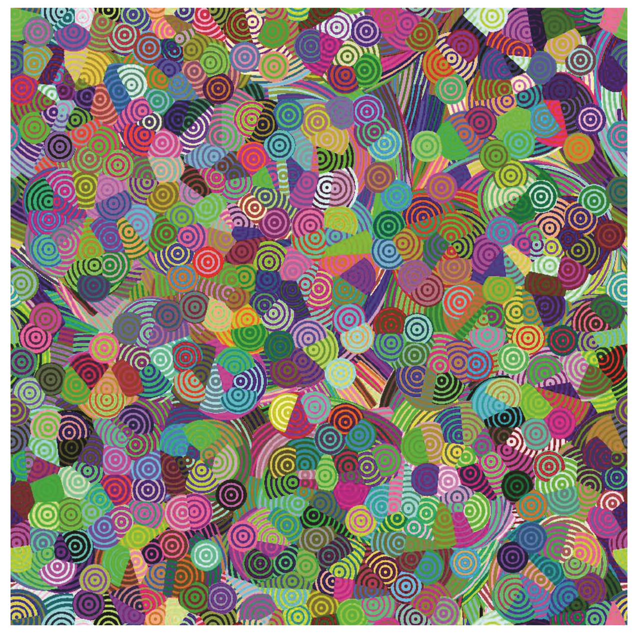


Figure 1: Stable marriage of Poisson and Lebesgue borrowed from [1].

Journal Publication

Haji-Mirsadeghi, M. O., and Khezeli, A. (2016). Stable transports between stationary random measures. *Electron. J. Prob.*, 21.

Some References

[1] Hoffman, C., Holroyd, A. E., and Peres, Y. (2006). A stable marriage of Poisson and Lebesgue. *Ann. Prob.*, 1241-1272.

[2] Last, G., and Thorisson, H. (2009). Invariant transports of stationary random measures and mass-stationarity. *Ann. Prob.*, 790-813.