Iteration of Boolean random varieties. Application to Fracture Statistics.

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Related publications

- Jeulin D. (2011) Multi-scale random sets: from morphology to effective properties and to fracture statistics, Proc. CMDS12 (Kolkata, 21-25-02-2011), J. Phys.: Conf. Ser. 319 012013
- Jeulin D. (2012) Morphology and effective properties of multi-scale random sets: A review, Comptes Rendus Mecanique, Comptes rendus de l'Académie des Sciences, Paris, doi:10.1016/j.crme.2012.02.004, Vol. 240 (4-5), pp. 219-229.
- Jeulin, D. (2015). Power Laws Variance Scaling of Boolean Random Varieties. Methodology and Computing in Applied Probability, 1-15.
- Jeulin, D. (2016) Iterated Boolean random varieties and application to fracture statistics models, Applications of Mathematics, 61(4), 363-386. DOI 10.1007/s10492-016-0137-7

Motivation

- Random sets showing long scale clustering effects: e.g. non-homogenous location of points in space, as seen for some defects in materials:
- In polycrystals modellled by random tessellations, defects located on the grain boundaries
- In composite materials, defects on fibers of a network
- In multivariate data analysis: clouds of points located on varieties
- Prediction of the fracture probability of specimens, for various generalizations of the Poisson point process and of the Boolean model, namely some **Cox processes and Cox Boolean models**

Outline

- Principle of random structure modeling
- Basic model of random sets: Boolean model
- Reminder on Poisson varieties
- Iteration of Boolean varieties
- Application to Fracture statistics models based on the weakest link model

Principle of random structure modeling

Characterization of a random structure: Main Criteria

- **Morphological criteria**
  - Size
  - Shape
  - Distribution in space (Clustering, Scales, Anisotropy)
  - Connectivity
- **Probabilistic criteria**
  - Probability laws (n points, sup_K)
  - Moments

Characterization of a random set

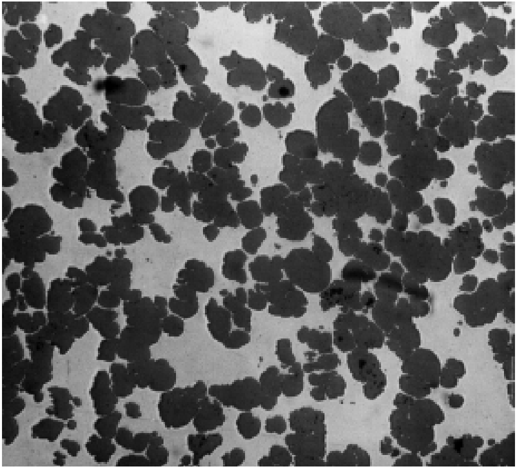
- Models derived from the **theory of Random Sets by G. MATHERON**
- For a random closed set A (RACS), characterization by the **CHOQUET capacity** $T(K)$ defined on the compact sets K

$$T(K) = P\{K \cap A \neq \emptyset\} = 1 - P\{K \subset A^c\} = 1 - Q(K)$$

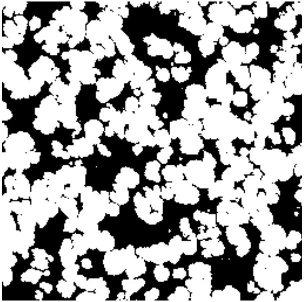
- In the Euclidean space $\mathbb{R}^n$, CHOQUET capacity and **dilation operation**

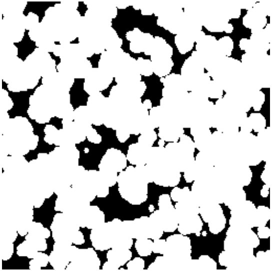
$$T(K_x) = P\{K_x \cap A \neq \emptyset\} = P\{x \in A \oplus \check{K}\}$$

Binary Morphology (Fe-Ag)

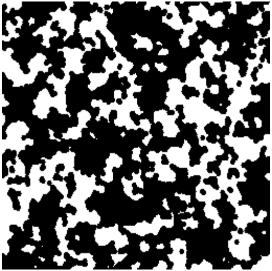


Basic Operations of Mathematical Morphology





Dilation by hexagon (2)



Erosion by hexagon (2)

Calculation of the CHOQUET capacity

For a given model, the functional T is obtained:

- by **theoretical calculation**
- by **estimation**
  - on simulations
  - on real structures (possible estimation of the parameters from the "experimental" T , and tests of the validity of assumptions).

Basic Models of Random Sets

Point Processes

Most simple kind of random structure: very small defects isolated in a matrix

- Particular RACS: Choquet capacity $T(K)$
- Probability generating function $G_K(s)$ of the random variable $N(K)$ (number of points of the process contained in K)
- Example: **Poisson point process** with intensity θ. Prototype random point process without any order

$$G_K(s) = \exp(\theta \mu_n(K)(s-1))$$

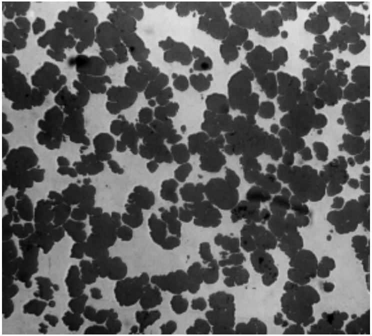
Boolean Model

- The Boolean model (G. Matheron, 1967) is obtained by implantation of random primary grains $A\prime$(with possible overlaps) on Poisson points x_k with the intensity θ:

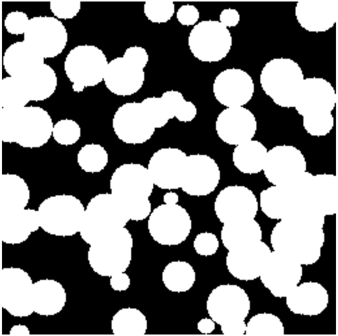
$$A = \cup_{x_k} A'_{x_k}$$

- Any shape (convex or non convex, and even non connected) can be used for the grain $A\prime$

Boolean Model

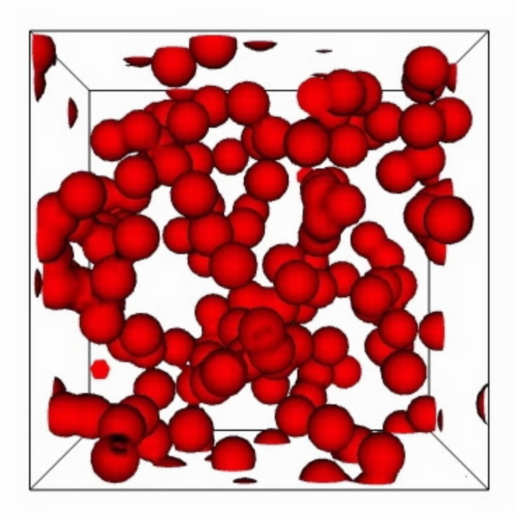


Fe Ag



Boolean model of spheres (0.5)

Boolean model of spheres



Boolean Model

Choquet capacity, with

$$q = P\{x \in A^c\}$$

$$\begin{aligned} T(K) &= P\{A \cap K \neq \emptyset\} = 1 - P\{K \subset A^c\} = 1 - \exp(-\theta\mu(A' \oplus \check{K})) \\ &= q^{\frac{\mu(A' \oplus \check{K})}{\mu(A')}} \end{aligned}$$

- Ex: contact distribution (ball), covariance, 3-points statistics,. . .
- Percolation threshold obtained from simulations: 0:2895 +- 0:0005 for spheres with a single diameter
- Estimation of the percolation threshold of the complementary random set of a boolean model of spheres (constant radius): 0.0540±0.005

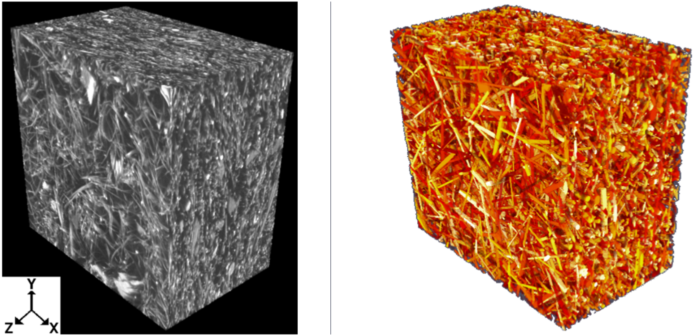
Long range random sets: Boolean varieties

Context

In some media, long fibre or extended strata networks, involving very long range of correlations .

- Needs to model and to simulate such media
- Slow convergence of the RVE

Context



Thermisorel fibrous network (X-ray microtomography and simulation)
600 x 600 x 360 voxels ; 5.6 x 5.6 x 3.37 mm3 ; Resolution US2B: 9.36 μm/voxel

Peyrega C., Jeulin D., Delisée C., Malvestio J. (2009) 3D morphological modelling of a random fibrous network. Image Analysis and Stereology.

Reminder on Poisson varieties

In $\mathbb{R}^n$, n Poisson linear varieties with dimension k ($k = 0, 1, ..., n-1$)

- Poisson point process $\{x_i(\omega)\}$, with intensity $\theta_k(d\omega)$ on the varieties of dimension $(n-k)$ containing the origin O, and with orientation ω
- On every point $x_i(\omega)$ is given a variety with dimension k, $V_k(\omega)_{x_i}$, orthogonal to the direction ω.

By construction, $V_k = \cup_{x_i(\omega)} V_k(\omega)_{x_i}$.
For instance in $\mathbb{R}^3$ can be built a network of Poisson hyperplanes Π_α (orthogonal to the lines D_ω containing the origin) or a network of Poisson lines in every plane Π_ω containing the origin.

Matheron G. (1975) Random sets and integral geometry.

Reminder on Poisson varieties

Theorem

The number of varieties of dimension k hit by a compact set K is a Poisson variable, with parameter $\theta(K)$:

$$\theta(K) = \int \theta_k(d\omega) \int_{K(\omega)} \theta_{n-k}(dx) = \int \theta_k(d\omega)\, \theta_{n-k}(K(\omega)) \qquad (1)$$

where $K(\omega)$ is the orthogonal projection of K on the orthogonal space to $V_k(\omega)$, $V_{k^\perp}(\omega)$. For the stationary case,

$$\theta(K) = \int \theta_k(d\omega)\, \mu_{n-k}(K(\omega)) \qquad (2)$$

Reminder on Poisson varieties

Theorem

The Choquet capacity $T(K) = P\{K \cap V_k \neq \varnothing\}$ *of the varieties of dimension* k *is given by*

$$T(K) = 1 - \exp\left(-\int \theta_k(d\omega) \int_{K(\omega)} \theta_{n-k}(dx)\right) \tag{3}$$

In the stationary case, the Choquet capacity is

$$T(K) = 1 - \exp\left(-\int \theta_k(d\omega)\ \mu_{n-k}(K(\omega))\right) \tag{4}$$

Reminder on Poisson varieties

Theorem

We consider now the isotropic (θ_k being constant) and stationary case, and a convex set K. Due to the symmetry of the isotropic version, we can consider $\theta_k(d\omega) = \theta_k\, d\omega$ as defined on the half unit sphere (in $\mathbb{R}^{k+1}$) of the directions of the varieties $V_k(\omega)$. The number of varieties of dimension k hit by a compact set K is a Poisson variable, with parameter $\theta(K)$ with:

$$\theta(K) = \theta_k \int \mu_{n-k}(K(\omega))\, d\omega = \theta_k \frac{b_{n-k} b_{k+1}}{b_n} \frac{k+1}{2} W_k(K) \qquad (5)$$

where b_k is the volume of the unit ball in $\mathbb{R}^k$ ($b_k = \dfrac{\pi^{k/2}}{\Gamma(1+\frac{k}{2})}$)

($b_1 = 2, b_2 = \pi, b_3 = \frac{4}{3}\pi$), and $W_k(K)$ is the Minkowski's functional of K, homogeneous and of degree $n-k$

Reminder on Poisson varieties

Following examples useful for applications:

- When $k = n - 1$, the varieties are **Poisson planes** in $\mathbb{R}^n$; in that case, $\theta(K) = \theta_{n-1} n W_{n-1}(K) = \theta_{n-1}\mathcal{A}(K)$, where $\mathcal{A}(K)$ is the norm of K (average projected length over orientations).
- In the plane $\mathbb{R}^2$ are obtained the **Poisson lines**, with $\theta(K) = \theta L(K)$, L being the perimeter.
- In the three-dimensional space are obtained **Poisson lines** for $k = 1$ and **Poisson planes** for $k = 2$. For Poisson lines, $\theta(K) = \frac{\pi}{4}\theta S(K)$ and for Poisson planes, $\theta(K) = \theta M(K)$, where S and M are the surface area and the integral of the mean curvature.

Boolean random varieties

Definition

A Boolean model with primary grain A' is built on Poisson linear varieties in two steps: i) we start from a network V_k; ii) every variety $V_{k\alpha}$ is dilated by an independent realization of the primary grain A'. The Boolean RACS A is given by

$$A = \cup_\alpha V_{k\alpha} \oplus A'$$

By construction, this model induces on every variety $V_{k^\perp}(\omega)$ orthogonal to $V_k(\omega)$ a standard Boolean model with dimension $n-k$ and with primary grain $A'(\omega)$ and with intensity $\theta(\omega)d\omega$.

Jeulin D. (1991) Modèles de Fonctions Aléatoires multivariables. Sci. Terre; 30: 225-256.

Boolean random varieties

Choquet capacity obtained by averaging over the directions ω; it can also be deduced from Eq. (4), after replacing K by $A' \oplus \check{K}$ and averaging.

Theorem

The Choquet capacity of the Boolean model built on Poisson linear varieties of dimension k is given by

$$T(K) = 1 - \exp\left(- \int \theta_k(d\omega)\, \overline{\mu}_{n-k}(A'(\omega) \oplus \check{K}(\omega)) \right) \tag{6}$$

For isotropic varieties, the Choquet capacity is given by

$$T(K) = 1 - \exp\left(-\theta_k \frac{b_{n-k} b_{k+1}}{b_n} \frac{k+1}{2} \overline{W}_k(A' \oplus \check{K}) \right) \tag{7}$$

Boolean random varieties

Particular cases of Eq. (6) when $K = \{x\}$ (giving the probability $q = P\{x \in A^c\} = \exp\left(-\int \theta_k(d\omega)\ \mu_{n-k}(A'(\omega))\right)$ and when $K = \{x, x+h\}$, giving the covariance of A^c, $Q(h)$:

$$Q(h) = q^2 \exp\left(\int \theta_k(d\omega)\ K_{n-k}(\omega, \overrightarrow{h}.\overrightarrow{u}(\omega))\right) \tag{8}$$

where $K_{n-k}(\omega, h) = \overline{\mu}_{n-k}(A'(\omega) \cap A'_{-h}(\omega))$ and $\overrightarrow{u}(\omega)$ is the unit vector with the direction ω.

For a compact primary grain A', there exists for any h an angular sector where $K_{n-k}(\omega, h) \neq 0$, so that the covariance generally does not reach its sill, at least in the isotropic case, and the **integral range is infinite.**

Jeulin D. (1991) Modèles de Fonctions Aléatoires multivariables. Sci. Terre; 30: 225-256.

Boolean model on Poisson planes

A Boolean model built on Poisson planes generates a structure with strata. On isotropic Poisson planes, we have for a convex set $A' \oplus \check{K}$ by application of equation (7):

$$T(K) = 1 - \exp\left(-\theta\,\overline{M}(A' \oplus \check{K})\right) \tag{9}$$

When A' and K are convex sets, $\overline{M}(A' \oplus \check{K}) = \overline{M}(A') + M(K)$. If $A' \oplus \check{K}$ is not convex, $T(K)$ is expressed as a function of the length l of the projection over the lines D_ω by

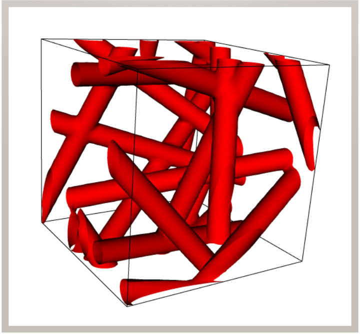
$$T(K) = 1 - \exp\left(-\theta \int_{2\pi ster} \bar{l}(A'(\omega) \oplus \check{K}(\omega)) \;\, d\omega\right).$$

If A' is a random sphere with a random radius R (with expectation $\overline{R}$) and K is a sphere with radius r, equation 9 becomes:

$$\begin{aligned} T(r) &= 1 - \exp -4\pi\theta(\overline{R} + r) \\ T(0) &= P\{x \in A\} = 1 - \exp -4\pi\theta\overline{R} \end{aligned}$$

which can be used to estimate θ and $\overline{R}$, and to validate the model.

Boolean model on Poisson lines in 3D



Isotropic Poison fibres

Faessel M., Jeulin D. (2011) 3D Multiscale Vectorial Simulation of Random Models, Proc. ICS 13, Beijing.

Schladitz K., Peters S., Reinel-Bitzer D., Wiegmann A., Ohser J., Design of acoustic trim based on geometric modeling and flow simulation for non-woven, Computational Materials Science 38 (2006) 56–66.

Boolean model on Poisson lines in 3D

A Boolean model built on Poisson lines generates a fiber network, with possible overlaps of fibers. On isotropic Poisson lines, for a convex set $A' \oplus \check{K}$

$$T(K) = 1 - \exp\left(-\theta \, \frac{\pi}{4} \overline{S}(A' \oplus \check{K})\right) \qquad (10)$$

If $A' \oplus \check{K}$ is not convex, $T(K)$ is expressed as a function of the area A of the projection over the planes Π_ω by

$$T(K) = 1 - \exp\left(-\theta \int_{4\pi ster} \overline{A}(A'(\omega) \oplus \check{K}(\omega)) \;\; d\omega\right) \qquad (11)$$

If A' is a random sphere with a random radius R (with expectation $\overline{R}$ and second moment $E(R^2)$) and K is a sphere with radius r, equation 10 becomes:

$$\begin{aligned} T(r) &= 1 - \exp\left(-\pi^2 \theta (E(R^2) + 2r\overline{R} + r^2)\right) \\ T(0) &= P\{x \in A\} = 1 - \exp\left(-\pi^2 \theta E(R^2)\right) \end{aligned}$$

which can be used to estimate θ, $E(R^2)$ and $\overline{R}$, and to validate the model.

Fluctuations and RVE of the volume fraction

Consider fluctuations of average values over different realizations of a random medium inside the domain B with the volume V. In Geostatistics, it is well known that for an ergodic stationary random function $Z(x)$, with mathematical expectation $E(Z)$, one can compute the variance $D_Z^2(V)$ of its average value $\bar{Z}(V)$ over the volume V as a function of the central covariance function $\overline{C}(h)$ of $Z(x)$ by :

$$D_Z^2(V) = \frac{1}{V^2} \int_B \int_B \overline{C}(x-y)\, dxdy, \tag{12}$$

where

$$\overline{C}(h) = E\{(Z(x) - E(Z))\,(Z(x+h) - E(Z))\}$$

Fluctuations and RVE of the volume fraction

For a large specimen (with $V \gg A_3$), equation (12) can be expressed to the first order in $1/V$ as a function of the integral range in the space R^3, A_3, by

$$D_Z^2(V) = D_Z^2 \frac{A_3}{V}, \quad (13)$$

$$\text{with } A_3 = \frac{1}{D_Z^2} \int_{R^3} \overline{C}(h)\, dh, \quad (14)$$

where D_Z^2 is the point variance of $Z(x)$ (here estimated on simulations) and A_3 is the integral range of the random function $Z(x)$, **defined when the integral in equations (12) and (14) is finite**.
When $Z(x)$ is the indicator function of the random set A, (13) provides the variance of the local volume fraction (in 3D) as a function of the point variance $D_Z^2 = p(1-p)$, p being the probability for a point x to belong to the random set A.

Scaling of the variance of the Boolean random varieties

Jeulin D. (2015) Power Laws Variance Scaling of Boolean Random Varieties, Methodology and Computing in Applied Probability, pp. 1-15.

Consider a convex domain K in $\mathbb{R}^n$, with Lebesgue measure $\mu_n(K)$

Scaling of the variance of the Boolean random varieties

Theorem

In $\mathbb{R}^n$, the variance $D_Z^2(K)$ of the local fraction $Z = \frac{\mu_n(A \cap K)}{\mu_n(K)}$ of a Boolean model built on isotropic Poisson varieties of dimension k ($k = 0, 1, ..., n-1$) V_k, is asymptotically expressed by

$$D_Z^2(K) = p(1-p)\left(\frac{A_k}{\mu_n(K)}\right)^{\frac{n-k}{n}}$$

the scaling exponent being $\gamma = \frac{n-k}{n}$. As particular cases, Poisson points ($k = 0$) give the standard Boolean model with a finite integral range and $\gamma = 1$, Poisson lines ($k = 1$) generate Poisson fibers with $\gamma = \frac{n-1}{n}$, and Poisson hyperplanes ($k = n-1$) provide Poisson strata with $\gamma = \frac{1}{n}$.

Poisson strata: very slow decrease of the variance with the volume of the sample K, with $\gamma = \frac{1}{n}$.

Scaling of the variance of the Boolean random varieties

- For **isotropic Boolean fibers in 2D**, the scaling exponent is $\gamma = \frac{1}{2}$
- For **isotropic Boolean fibers in 3D**, $\gamma = \frac{2}{3}$. Exponent recovered from numerical simulations for the volume fraction and for the effective elastic properties [1]. For fibers with a finite length, an intermediary situation will occur [2], and a scaling coefficient $\frac{2}{3} \leqslant \gamma \leqslant 1$, depending on the size of the specimen is expected ($\gamma \simeq \frac{2}{3}$ for small specimens, and $\gamma \simeq 1$ for large samples)

Dirrenberger J., Forest S., Jeulin D. (2014) Towards gigantic RVE sizes for 3D stochastic fibrous networks, International Journal of Solids and Structures, volume 51(2), 2014, pp. 359-376.

Altendorf H., Jeulin D., Willot F. (2014) Influence of fiber geometry on the effective properties: application to glassfiber-reinforced composites, Influence of the fiber geometry on the macroscopic elastic and thermal properties, International Journal of Solids and Structures, Vol. 51, pp. 3807-3822.

Scaling of the variance of the Boolean random varieties

- For **isotropic Boolean strata in 3D**, $\gamma = \frac{1}{3}$. Decrease of the variance with size much slower than for a finite integral range.

Iteration of Boolean random varieties

Two steps Boolean varieties

Further generalization of Boolean models by iteration of Poisson varieties

- For instance in $\mathbb{R}^2$, first consider Poisson lines, and in a second step Poisson points on every line. Points are germs to locate primary grains A' to generate a Boolean model. Compared to the standard Boolean model, alignments of grains
- Similarly in $\mathbb{R}^3$, start from Poisson planes $V_{2\alpha}$ and use Poisson lines $V_{1\beta}$ in every plane to generate a Boolean model with fibers. In contrast with Poisson fibers in $\mathbb{R}^3$, this model generates a random set with some coplanar fibers: could mimic specific microstructures with an order in a lower dimension subspace of $\mathbb{R}^n$, such as preferred germination of objects on specific planes or lines.

Models based on **doubly stochastic Poisson random variables** for which the Choquet capacity can be obtained

Two steps Boolean varieties

Definition

Two steps random varieties are defined as follows: starting from Poisson linear varieties V_k of dimension k and with intensity $\theta_k(d\omega)$ in $\mathbb{R}^n$, Poisson linear varieties $V_{k'\beta}$ with dimension $0 \leq k' < k$ and with intensity $\theta_{k'}(d\omega)$ are implanted on each $V_{k\alpha}$. Then each $V_{k'\beta}$ is dilated by independent realizations of a random compact primary grain $A' \subset \mathbb{R}^n$ to generate the Boolean RACS A:

$$A = \cup_\beta V_{k'\beta} \oplus A'$$

By construction, when $k' = 0$ the varieties $V_{k'\beta}$ are a particular case of a Cox process driven by the random set V_k, and the derived random set A is a Cox Boolean model (Jeulin 2012).
In what follows, purpose restricted to the stationary and isotropic case, with the two intensities θ_k and $\theta_{k'}$

Two steps Boolean varieties: Generating function

Theorem

The number $N(K)$ of varieties of dimension $k' < k$ hit by the compact set K is a random variable with generating function

$$G_{k'}(s,K) = E\{s^{N(K)}\} = \exp\left[\theta_k a_k W_k(K)\left[\varphi_{k'}(\theta_{k'} a_{k'}(1-s),K) - 1\right]\right] \tag{15}$$

where $a_{k'} = \frac{b_{n-k'} b_{k'+1}}{b_n} \frac{k'+1}{2}$ and $\varphi_{k'}(\lambda, K)$ is the Laplace transform of the random variable $W_{k'}(K \cap V_{k\alpha})$, $W_{k'}$ being the Minkowski functional homogeneous with degree $k - k'$ in $\mathbb{R}^k$:

$$\varphi_{k'}(\lambda, K) = E\{\exp\left[-\lambda W_{k'}(K \cap V_k)\right]\} \tag{16}$$

the mathematical expectation being taken over the realizations $V_{k\alpha}$.

Two steps Boolean varieties: Choquet capacity

Theorem

As a consequence, the Choquet capacity of the Boolean RACS A built on the Poisson linear varieties $V_{k'}$ using a deterministic primary grain A' is derived from $G_{k'}(0, A' \oplus \check{K})$, $E\{\}$ being the expectation with respect to the random variety $V_{k\alpha}$:

$$
\begin{aligned}
&1 - T(K) \qquad (17)\\
&= \exp\left[-\theta_k a_k W_k(A' \oplus \check{K})\left[1 - E\{\exp\left[-\theta_{k'} a_{k'} W_{k'}(A' \oplus \check{K} \cap V_k)\right]\}\right]\right]
\end{aligned}
$$

Choquet capacity requires the use of the Laplace transform $\varphi_{k'}(\lambda, A' \oplus \check{K})$ Not easy to express them in a closed form for specific compact sets K and A', but can be estimated by simulation of the random variables obtained from random variables $W_{k'}(A' \oplus \check{K} \cap V_{k\alpha})$ obtained from random sections Examples of closed form expressions given now for two-steps Poisson points in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Scaling of the variance of two steps Boolean random varieties

- The local volume fraction of A, $\frac{\mu_n(A \cap K)}{\mu_n(K)}$ has a variance proportional to $\frac{1}{\mu_n(K)^{\frac{n-k}{n}}}$
- Scaling law of the variance : **same as for Boolean varieties, the variance of the volume fraction of iterated varieties being dominated by the effect of the varieties of the first iteration**, namely V_k.
- In $\mathbb{R}^2$ a Boolean model built on **Poisson lines or on Poisson points generated on Poisson lines** have the same scaling with the exponent $\gamma = \frac{1}{2}$
- In $\mathbb{R}^3$ a Boolean model built on **Poisson lines or on Poisson points in Poisson planes** have the same scaling with the exponent $\gamma = \frac{1}{3}$, while for a Boolean model built on Poisson fibers or on **Poisson points located on Poisson lines** we have $\gamma = \frac{2}{3}$

Two steps Boolean varieties in 2D and in 3D

- Poisson points on Poisson lines in $\mathbb{R}^2$
- Poisson points on Poisson planes in $\mathbb{R}^3$
- Poisson points on Poisson lines in $\mathbb{R}^3$

Poisson points on Poisson lines in 2D

Two steps point process:

1. Poisson lines in $\mathbb{R}^2$ (isotropic case), with intensity θ_1
2. On each Poisson line, 1D Poisson point process, with intensity θ

Poisson points on Poisson lines in 2D

Theorem

The generating function $G_K(s)$ of the random number of points $N_P(K)$ contained in a convex set K in $\mathbb{R}^2$ with perimeter $\mathcal{L}(K)$, random intercept length $L(K)$ (with Laplace transform $\varphi_L(\lambda, K)$), is given by

$$G_K(s) = \exp\left(-\theta_1 \mathcal{L}(K)(1 - \varphi_L(\theta(1-s), K))\right) \tag{18}$$

We have

$$1 - T(K) = Q(K) = \exp\left(-\theta_1 \mathcal{L}(K)(1 - \varphi_L(\theta, K))\right) \tag{19}$$

Choquet capacity of the corresponding Boolean model for convex sets K and A' obtained by replacing K by $(A' \oplus \check{K})$ in equation 19

When K is the disc $C(r)$ with radius r, generating function of the random number of points $N_P(r)$ in $C(r)$

$$G(s,r) = \exp\left[-2\pi r \theta_1 (1 - \varphi_L(\theta(1-s), r))\right]$$

with

$$\varphi_L(\lambda, r) = \pi/2 \left[-BesselI(1, 2\lambda r) + StruveL(-1, 2\lambda r)\right]$$

Choquet capacity:

$$1 - T(r) = Q(r) = \exp\left[-2\pi r \theta_1 (1 - \varphi_L(\theta, r))\right]$$

Poisson points on Poisson planes in 3D

Two steps point process:

1. Poisson planes in $\mathbb{R}^3$ (isotropic case), with intensity θ_2
2. On each Poisson plane, 2D Poisson point process, with intensity θ

For a convex compact set K,

$$G_K(s) = \exp\left[\theta_2 M(K)(\psi_A(\theta(1-s), K \cap \pi) - 1)\right]$$

$$1 - T(K) = Q(K) = \exp\left[-\theta_2 M(K)(1 - \psi_A(\theta, K \cap \pi))\right]$$

with:

- $M(K)$: integral of mean curvature of K
- $A(K \cap \pi)$: area of sections of K by a random plane π, with Laplace transform $\psi_A(\lambda, K \cap \pi)$

Generating function $G(s, r)$ of the random number of points $N_P(r)$ in the sphere with radius r

$$G(s, r) = \exp\left[-4\pi r\theta_2(1 - \psi(\theta\pi(1 - s), r))\right]$$

$$1 - T(r) = Q(r) = \exp\left[-4\pi r\theta_2(1 - \psi(\theta\pi, r))\right]$$

with

$$\psi(\lambda, r) := \frac{\exp(-\lambda r^2) \int_0^{r\sqrt{\lambda}} \exp(y^2) dy}{r\sqrt{\lambda}}$$

Poisson points on Poisson lines in 3D

Two steps point process:

1. Poisson lines in $\mathbb{R}^3$ (isotropic case), with intensity θ_1
2. On each Poisson line, 1D Poisson point process, with intensity θ

Models of a long fiber network with points germs

Poisson points on Poisson lines in 3D

Generating function and Choquet capacity

For a convex compact set K,

$$G_K(s) = \exp\left[-\frac{\pi}{4}\theta_1 S(K)(1-\varphi_L(\theta(1-s),K))\right]$$

$$1 - T(K) = Q(K) = \exp\left[-\frac{\pi}{4}\theta_1 S(K)(1-\varphi_L(\theta,K))\right]$$

where $S(K)$ is the surface area of K, and $\varphi_L(\lambda, K)$ the Laplace transform of a random chord $L(K)$ in K

Generating function $G(s,r)$ of the random number of points $N_P(r)$ in the sphere with radius r

$$\begin{aligned}\log(G(s,r)) &= \\ & -\pi^2\theta_1 r^2\left(1-\frac{2}{(2r\theta(1-s))^2}\right. \\ & \left.[1-(1+2r\theta(1-s))\exp(-2r\theta(1-s))]\right)\end{aligned}$$

$$\begin{aligned}1-T(r) &= Q(r) = \\ & \exp\left[-\pi^2\theta_1 r^2\left(1-\frac{2}{(2r\theta)^2}[1-(1+2r\theta)\exp(-2r\theta)]\right)\right]\end{aligned}$$

Three steps iteration: Poisson points on 2D Poisson lines on 3D Poisson planes

1. Poisson planes in $\mathbb{R}^3$ (isotropic case), with intensity θ_2
2. On each Poisson plane, 2D Poisson lines process, with intensity θ_1
3. On each line, 1D Poisson point process with intensity θ

Models of a long fibers in random planes, with point defects

Three steps iteration: Poisson points on 2D Poisson lines on 3D Poisson planes

Generating function and Choquet capacity

Consider a convex compact set K, with random planar sections $K \cap \pi$

$$\begin{aligned} & \log(G_K(s)) \\ = \; & \theta_2 M(K)(E_\pi \left\{\exp\left[\theta_1 \mathcal{L}(K \cap \pi)\left(\varphi_L(\theta(1-s), K \cap \pi) - 1\right)\right]\right\} - 1) \end{aligned}$$

where E_π is the mathematical expectation over random sections

$$\begin{aligned} \log(Q(K)) \; = & \\ & \theta_2 M(K)(E_\pi \left\{\exp\left[\theta_1 \mathcal{L}(K \cap \pi)\left(\varphi_L(\theta, K \cap \pi) - 1\right)\right]\right\} - 1) \end{aligned}$$

with

- perimeter $\mathcal{L}(K \cap \pi)$ of sections of K, with Laplace transform $\psi_{\mathcal{L}}(\lambda, K \cap \pi)$,
- random chord of each planar section $L(K \cap \pi)$, with Laplace transform $\varphi_L(\lambda, K \cap \pi)$

Three steps iteration: Poisson points on 2D Poisson lines on 3D Poisson planes

Generating function for a sphere

Generating function of the number of points of the process inside a sphere with radius r

$$\begin{aligned} & \log(G(s,r)) \\ = \; & 4\pi\theta_2 r (E_R\left\{\exp\left[2\pi\theta_1 R\left(\varphi_L(\theta(1-s),R)-1\right)\right]\right\}-1) \end{aligned}$$

with

$$\begin{aligned} & E_R\left\{\exp\left[2\pi\theta_1 R\left(\varphi_L(\theta(1-s),R)-1\right)\right]\right\} \\ = \; & \int_0^r \exp\left[2\pi\theta_1 u\left(\varphi_L(\theta(1-s),u)-1\right)\right] f(u,r)du \end{aligned}$$

- $f(u,r)$: distribution function of the radius R of random sections of a sphere
- $\varphi_L(\lambda,R)$: Laplace transform of random chords of the disc with radius R

Three steps iteration: Poisson points on 2D Poisson lines on 3D Poisson planes

Choquet capacity for a sphere

$$\begin{aligned}\log(Q(r)) &= \log(1 - T(r)) = \\ & 4\pi\theta_2 r \left(\int_0^r \exp\left[2\pi\theta_1 u \left(\varphi_L(\theta, u) - 1\right)\right] f(u, r) du - 1 \right)\end{aligned}$$

Fracture Statistics Models

Recall of the weakest link model

- Decomposition of the volume V into links v_i
- Fracture of the volume V when a single link v_i is broken

Classical computation for independent events

$$P\{\text{Non fracture}\} = \prod_i P\{\text{Non fracture of } v_i\}$$

For $v_i \to 0$, $P\{\text{fracture}\} \simeq \Phi((\sigma(x))dx$, Φ increasing with the loading, and $P\{\text{non fracture of } dx\} \simeq 1 - \Phi((\sigma(x))dx$

$$P\{\text{Non fracture of } V\} = \exp(-\int_V \Phi(\sigma(x))dx) = \exp(-V\Phi(\sigma_{eq}))$$

where the equivalent stress σ_{eq} is defined by

$$\Phi(\sigma_{eq}) = \frac{1}{V}\int_V \Phi((\sigma(x))dx$$

Recall of the weakest link model

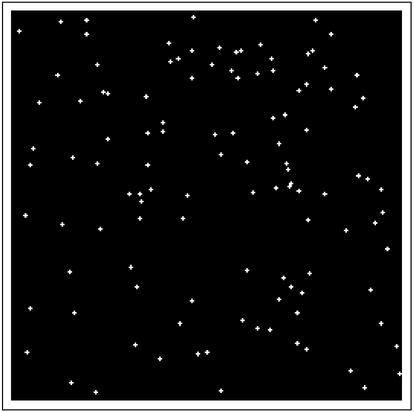
- Critical Defects at level σ: **Poisson point process** (Prototype **random** process without any order) with the cumulative intensity $\Phi(\sigma)$ (total number of point defects per unit volume, with critical stress less than σ) in a matrix with $\sigma_R = \infty$
- Model: $\wedge$ **Boolean random function** with point support primary random functions (PRF); immediate extension to any PRF with a support having almost surely compact sections.

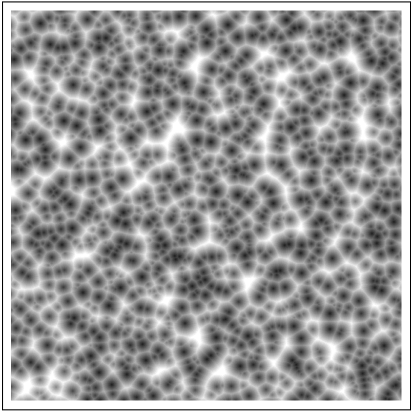
For a homogeneous stress field $\sigma(x) = \sigma$

$$P\{\text{Non fracture of } V\} = \exp(-V\Phi(\sigma))$$

For $\Phi(\sigma) = \theta(\sigma - \sigma_0)^m$, $P\{\text{Non fracture of } V\}$ follows a Weibull distribution

Poisson point process in the plane and Inf Boolean random function with cone PRF





Point defects generated by iteration of random Poisson varieties

- Poisson points on Poisson planes in $\mathbb{R}^3$
- Poisson points on Poisson lines in $\mathbb{R}^3$
- Three steps iteration: Poisson points on 2D Poisson lines on Poisson planes in $\mathbb{R}^3$

Poisson points on Poisson planes

Two steps point process:

1. Poisson planes in $\mathbb{R}^3$ (isotropic case), with intensity θ_2
2. On each Poisson plane, 2D Poisson point process, with intensity θ

In the case of point defects acting in fracture statistics, θ replaced by $\Phi(\sigma)$
Considering the Poisson tessellation generated by Poisson planes, this model figures out **point defects located on grain boundaries, generating intergranular fracture**

$$P\{\sigma_R \geq \sigma\}_\pi = \exp\left[-\theta_2 M(K)(1 - \psi_A(\Phi(\sigma), K \cap \pi))\right] \quad (20)$$

When K is the sphere of radius r :

$$P\{\sigma_R \geq \sigma\}_\pi = \exp\left[-4\pi r \theta_2 (1 - \psi(\pi\Phi(\sigma), r))\right] \quad (21)$$

with

$$\psi(\lambda, r) := \frac{\exp(-\lambda r^2) \int_0^{r\sqrt{\lambda}} \exp(y^2) dy}{r\sqrt{\lambda}}$$

Poisson points on Poisson lines

Two steps point process:

1. Poisson lines in $\mathbb{R}^3$ (isotropic case), with intensity θ_1
2. On each Poisson line, 1D Poisson point process, with intensity θ

Models of a long fiber network with point defects. In the case of point defects acting in fracture statistics, θ replaced by $\Phi(\sigma)$

$$P\{\sigma_R \geq \sigma\}_D = \exp\left[-\frac{\pi}{4}\theta_1 S(K)(1 - \varphi_L(\Phi(\sigma), K))\right] \qquad (22)$$

Poisson points on Poisson lines

When K is the sphere with radius r

$$P\{\sigma_R \geq \sigma\}_D = \exp\left[-\pi^2\theta_1 r^2 (1 - \frac{2}{(2r\Phi(\sigma))^2} [1 - (1 + 2r\Phi(\sigma))\exp(-2r\Phi(\sigma))]\right] \tag{23}$$

Comparison of Fracture statistics for standard Poisson points and for points on planes

Fracture of a sphere of sphere of radius r containing a random number of points $N_P(r)$ with a given average:
For the standard Poisson point process,

$$E\{N_P(r)\} = \frac{4}{3}\pi r^3 \theta_3$$

For Poisson points on Poisson planes,

$$E\{N_P(r)\} = \frac{8}{3}\pi^2 r^3 \theta_2 \theta$$

Given an average number of defects in the sphere of radius r,

$$\theta_3 = 2\pi\theta_2\theta$$

Comparison of Fracture statistics for standard Poisson points and for points on planes

Using the same intensity $\Phi(\sigma) = \theta$ for the two processes

$$\begin{aligned}\log(P\{\sigma_R &\geq \sigma\}_P) - \log(P\{\sigma_R \geq \sigma\}_\pi) \\ &= 4\pi r\theta_2 \left(1 - \psi(\theta\pi, r) - \frac{\theta r^2}{2}\right)\end{aligned}$$

$$P\{\sigma_R \geq \sigma\}_\pi < P\{\sigma_R \geq \sigma\}_P \text{ for } r^2\Phi(\sigma) < 1.8$$

Given the same statistics of defects $\Phi(\sigma)$, **for low applied stresses, or at a small scale,** the **"intergranular" fracture probability is higher than the standard probability of fracture**. For **high applied stresses** the reverse is true, and the **material is less sensitive to "intergranular" fracture**

Crossing of the two probability curves for $r^2\Phi(\sigma) \simeq 1.8$

Comparison of Fracture statistics for standard Poisson points and for points on lines

Fracture of a sphere of sphere of radius r containing a random number of points $N_P(r)$ with a given average:
For the standard Poisson point process,

$$E\{N_P(r)\} = \frac{4}{3}\pi r^3 \theta_3$$

For Poisson points on Poisson lines,

$$E\{N_P(r)\} = \frac{4}{3}\pi^2 r^3 \theta_1 \theta$$

Given an average number of defects in the sphere of radius r,

$$\theta_3 = \pi \theta_1 \theta$$

Comparison of Fracture statistics for standard Poisson points and for points on lines

Using the same intensity $\Phi(\sigma) = \theta$ for the two processes

$$\begin{aligned}\log(P\{\sigma_R &\geq \sigma\}_D) - \log(P\{\sigma_R \geq \sigma\}_P) \\ &= \pi^2 \theta_1 r^2 \left(\frac{2}{\alpha^2}(1 - (1+\alpha)\exp(-\alpha)) - 1 + \frac{2}{3}\alpha \right)\end{aligned}$$

with $\alpha = 2r\theta$

$$P\{\sigma_R \geq \sigma\}_D < P\{\sigma_R \geq \sigma\}_P \text{ for any } \alpha$$

Given the same statistics of defects $\Phi(\sigma)$, the **"fiber" fracture probability is higher than the standard probability of fracture**. The **material is more sensitive to points defects on fibers**

Comparison of Fracture statistics for Poisson points on planes and for points on lines

Given an average number of defects in the sphere of radius r,

$$\theta_2 \theta_\pi = \frac{1}{2} \theta_1 \theta_D$$

Taking $2r\theta_D = \pi r^2 \theta_\pi = \alpha$, we get $\pi^2 \theta_1 r^2 = 4\pi r \theta_2$
Using the same intensity $\Phi(\sigma) = \theta = \theta_\pi = \theta_D$ for the two processes,

$$\begin{aligned} \log(P\{\sigma_R &\geq \sigma\}_\pi) - \log(P\{\sigma_R \geq \sigma\}_D) \\ &= \pi^2 \theta_1 r^2 \left(\psi(\theta\pi, r) - \frac{2}{\alpha^2}(1 - (1+\alpha)\exp(-\alpha)) \right) > 0 \end{aligned}$$

and therefore $P\{\sigma_R \geq \sigma\}_\pi > P\{\sigma_R \geq \sigma\}_D$ and **it is easier to break a specimen with defects on fibers than with defects on planes**.

Three steps iteration: Poisson points on 2D Poisson lines on Poisson planes

1. Poisson planes in $\mathbb{R}^3$ (isotropic case), with intensity θ_2
2. On each Poisson plane, 2D Poisson lines process, with intensity θ_1
3. On each line, 1D Poisson point process with intensity θ

Models of a long fibers in random planes, with point defects
In the case of point defects acting in fracture statistics, θ replaced by $\Phi(\sigma)$

Three steps iteration: Poisson points on 2D Poisson lines on Poisson planes

$$\log(P\{\sigma_R \geq \sigma\}_{3\text{ iterations}}) = \theta_2 M(K) \left(E_\pi \left\{\exp\left[\theta_1 \mathcal{L}(K\cap\pi)\left(\varphi_L(\Phi(\sigma), K\cap\pi) - 1\right)\right]\right\} - 1\right)$$

For fracture statistics of the sphere with radius r,

$$\log(P\{\sigma_R \geq \sigma\}_{3\text{ iterations}}) = 4\pi\theta_2 r \left(\int_0^r \exp\left[2\pi\theta_1 u\left(\varphi_L(\Phi(\sigma), u) - 1\right)\right] f(u,r)du - 1\right)$$

Comparison of Fracture statistics for Poisson points and for the three steps iteration

Fracture of a sphere of radius r containing a random number of points $N_P(r)$ with a given average:
For the standard Poisson point process,

$$E\{N_P(r)\} = \frac{4}{3}\pi r^3 \theta_3$$

For Poisson points on Poisson lines on Poisson planes,

$$E\{N_P(r)\} = \frac{4}{3}\pi r^3 (\theta_2 \theta_1 \theta 2\pi^2)$$

Given an average number of defects in the sphere of radius r,

$$\theta_3 = 2\pi^2 \theta_2 \theta_1 \theta$$

To compare fracture statistics of Poisson points and of the three iterations case, use of the ratio

$$\frac{4}{3}\frac{\theta_3 r^3}{4\theta_2 r} = \frac{4}{3}\frac{2\pi^2\theta_2\theta_1\theta}{4\theta_2} r^2 = \frac{2}{3}\pi^2\theta_1\theta r^2$$

Comparison of Fracture statistics for Poisson points and for the three steps iteration

With auxilliary variables $2\theta r = \alpha$ and $\theta_1 r = \beta$, we have to compare $\frac{1}{3}\pi^2\theta_1\alpha r = \frac{1}{3}\pi^2\alpha\beta$ to

$$1 - \int_0^1 \exp\left[-2\pi\beta y\left(1 - \varphi_L(\theta, ry)\right)\right] \frac{y}{\sqrt{1-y^2}} dy$$

Comparison by numerical calculation of the integral, made over α, for given β. For $\beta = 0.01,\ 0.1$, 1 and 10,

$$P\{\sigma_R \geq \sigma\}_{3 \text{ iterations}} > P\{\sigma_R \geq \sigma\}_P$$

With these assumptions, the **strength of a medium with Poisson point defects is lower than for the defects localized on lines in planes**

Comparison of Fracture statistics for Poisson points on Poisson planes and for the three steps iteration

Average number of points in the sphere with radius r
For Poisson points on Poisson planes

$$E\{N_\pi(r)\}\frac{8}{3}\pi^2 r^3 \theta_2 \theta_\pi$$

and for 3 iterations

$$E\{N_P(r)\}_3 = \frac{4}{3}\pi r^3 (\theta_2 \theta_1 \theta 2\pi^2)$$

To keep the same average values,

$$2\theta_2\theta_\pi = 2\pi\theta_2\theta_1\theta$$

Taking $\theta_\pi = \theta$ to get the same statistics over points, and θ_2 identical for the two models, in order to keep the same scale for the Poisson polyedra, $\pi\theta_1 = 1$ and $\theta_1 = 1/\pi$

Comparison of Fracture statistics for Poisson points on Poisson planes and for the three steps iteration

For $\beta = 0.01, 0.1, 0.5, 0.75$, numerical calculations give

$$P\{\sigma_R \geq \sigma\}_{3 \text{ iterations}} > P\{\sigma_R \geq \sigma\}_{\pi}$$

For $\beta = 1$,

$$P\{\sigma_R \geq \sigma\}_{3 \text{ iterations}} < P\{\sigma_R \geq \sigma\}_{\pi} \text{ when } \alpha < 1.99$$
$$P\{\sigma_R \geq \sigma\}_{3 \text{ iterations}} > P\{\sigma_R \geq \sigma\}_{\pi} \text{ when } \alpha > 1.99$$

For $\beta = 2, 10$,

$$P\{\sigma_R \geq \sigma\}_{3 \text{ iterations}} < P\{\sigma_R \geq \sigma\}_{\pi}$$

Conclusion

- New models of random sets and point processes designed to simulate some **specific clustering of points**, namely on random lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ and on random planes in $\mathbb{R}^3$
- Possible application is to **model point defects in materials with some degree of large scale alignments**
- Derivation of general theoretical results, useful to compare geometrical effects on the sensitivity of materials to fracture
- Easy **generalization to more than one critical defect**, using the distribution of the number of defects in a domain
- Based on the presented theoretical results, applications can be looked for from statistical experimental data.