Limit Theorems for Multidimensional Renewal Sets

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The talk is based on joint work with Ilya Molchanov (Bern, Switzerland).

Introduction

Let $\left(\xi_m, m \in \mathbb{N}^d\right)$, $d \geq 2$, be a multi-indexed family of i.i.d. random variables with finite common mean $\mu > 0$.

Denote by S_n, $n \in \mathbb{N}^d$, their partial sums over rectangles:

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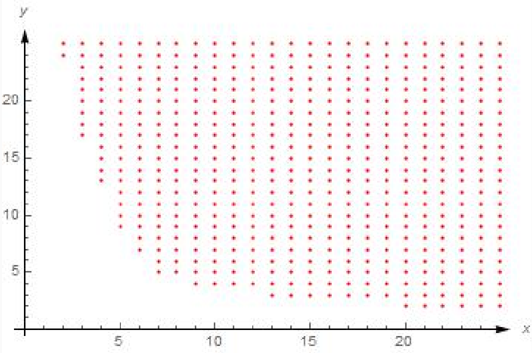
In the multidimensional case, the latter is not applicable anymore due to the lack of natural total order in $\mathbb{N}^d$. So, a relevant multidimensional renewal process should be set-valued!

For $t > 0$, consider the renewal sets $\mathcal{M}_t$ of the following form:

$$\mathcal{M}_t = \{n \in \mathbb{N}^d \colon S_n \geq t\}.$$



Two basic questions:

1) How large are $\mathcal{M}_t$? (Or, more precisely, how large are $\overline{\mathcal{M}}_t$?)
2) What do $\mathcal{M}_t$ look like?

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Answer to 1):
asymptotic results on the cardinality of $\overline{\mathcal{M}}_t$. Some limit theorems are summarized in the recent monograph by Klesov (2014).

Theorem (SLLN for card $\overline{\mathcal{M}}_t$, O. Klesov, 1991)

$$\begin{cases} \xi \geq 0 \ a.s., \\ \mathbb{E}\Big(\xi \log_+^{d-1} \xi\Big) < \infty, \end{cases} \implies \lim_{t\to\infty} \frac{\operatorname{card} \overline{\mathcal{M}}_t}{t \log^{d-1} t} = \frac{1}{\mu(d-1)!} \ a.s.$$

The same asymptotics holds for the renewal function $U(t) = \mathbb{E} \operatorname{card} \overline{\mathcal{M}}_t$.

Theorem (Marcinkiewicz-Zygmund SLLN for card $\overline{\mathcal{M}}_t$, O. Klesov, J. Steinebach, 1997)

Let

i) $\xi \geq 0$ *a.s.,*

ii) $\mathbb{E}\Big(\xi^\beta \log_+^{d-1} \xi\Big) < \infty$ *for* $\beta < \beta_0(d)$ *with some* $\beta_0(d) \in [1, 2]$.

Then

$$\lim_{t\to\infty} \frac{\operatorname{card} \overline{\mathcal{M}}_t - \frac{t}{\mu}\mathcal{P}(\log \frac{t}{\mu})}{t^{1/\beta} \log^{d-1} t} = 0 \text{ a.s.}$$

Here $\mathcal{P}$ is a polynomial of degree $d - 1$ which can be explicitly given.

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Answer to 2):

asymptotic results on the location and the shape of $\mathcal{M}_t$
(the aim of the talk).

An informal discussion

We start with the SLLN for the multi-indexed case.

Standing notation: $|n| \stackrel{\text{def}}{=} n_1 \cdot \ldots \cdot n_d$.

Theorem (multi-indexed SLLN, R. Smythe, 1973)

$$\mathbb{E}\Big(|\xi| \log_+^{d-1} |\xi|\Big) < \infty \iff \lim_{|n|\to\infty} \frac{S_n}{|n|} = \mu \text{ a.s.}$$

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$$\mathbb{E}\Big(|\xi| \log_+^{d-1} |\xi|\Big) < \infty \Longleftrightarrow \lim_{|n|\to\infty} \frac{S_n}{|n|} = \mu \ a.s.$$

In other words, $S_n \approx \mu|n|$.

Thus, $S_n \geq t$ roughly means that $|n| \geq \mu^{-1}t$.

Equivalently,

$$t^{-1/d} \cdot \{n \colon S_n \geq t\} \approx \ t^{-1/d} \cdot \{n \colon |n| \geq \mu^{-1}t\}.$$

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Equivalently,

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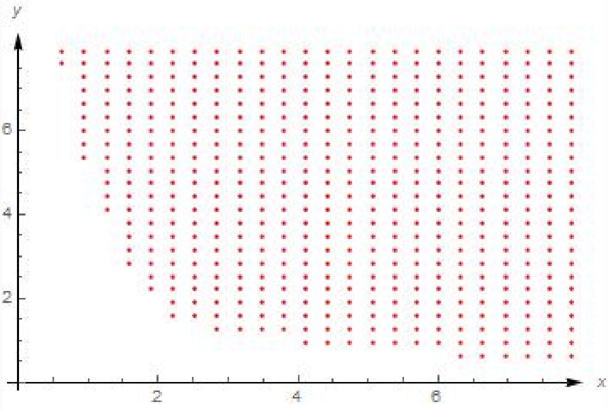
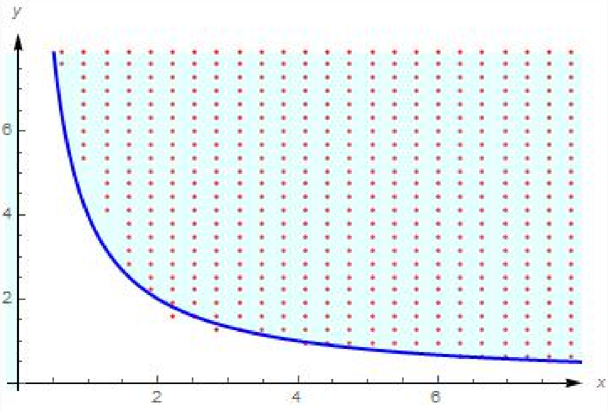
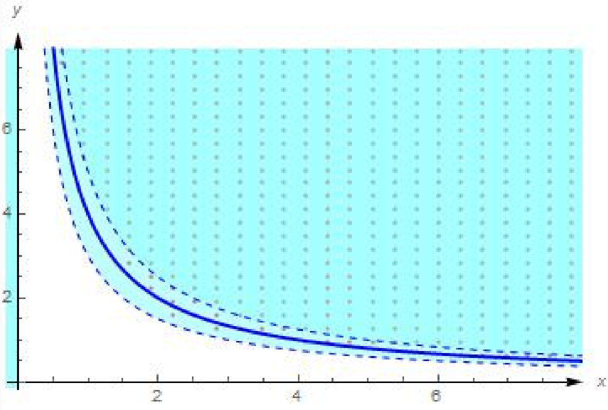
The rescaled renewal set $t^{-1/d}\mathcal{M}_t$



The rescaled renewal set $t^{-1/d}\mathcal{M}_t$, the "limit" set $\mathcal{H}$



The rescaled renewal set $t^{-1/d}\mathcal{M}_t$, the "limit" set $\mathcal{H}$, and its inner and outer neighbourhoods.



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Let $|n|$ stand for $n_1 \cdot \ldots \cdot n_d$.

Theorem (multi-indexed SLLN, R. Smythe, 1973)

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In other words, $S_n \approx \mu|n|$.
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Equivalently,

$$\underbrace{t^{-\frac{1}{d}} \cdot \{n \colon S_n \geq t\}}_{t^{-1/d}\mathcal{M}_t \text{ (rescaled } \mathcal{M}_t)} \overset{?}{\approx} \underbrace{t^{-\frac{1}{d}} \cdot \{n \colon |n| \geq \mu^{-1} t\}}_{\substack{\text{approximate the set} \\ \mathcal{H}=\left\{x\in\mathbb{R}_+^d \colon |x|\geq\mu^{-1}\right\} \text{ as } t\to\infty}}.$$

For $c \in \mathbb{R}$, let us introduce the neighbourhoods of $\mathcal{H}$:

$$\mathcal{H}(c) = \left\{x \in \mathbb{R}_+^d \colon |x| \geq \mu^{-1} + c\right\}.$$

Notice that $\mathcal{H}(c)$ decreases in c and $\mathcal{H}(0) = \mathcal{H}$.

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In the rest of the talk, we discuss how close $t^{-1/d}\mathcal{M}_t$ and $\mathcal{H}$ are. We use two different approaches:

1) **set-inclusion**
   (bounds in terms of set inclusions for $t^{-1/d}\mathcal{M}_t$ and $\mathcal{H}(c)$);
2) **metrical**
   (bounds in terms of distances between sets).

A problem with the metrical approach in case of lattice sets: we have to extremely accurately count the number of integer points between "hyperbolas" $\{x\colon |x| = c_1\}$ and $\{x\colon |x| = c_2\}$. This is closely related to some number-theoretic issues (the generalized Dirichlet divisor problem). The required bounds are only conjectured and go back to the Riemann Hypothesis.

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A way out is to use continuous counterparts of $\mathcal{M}_t$ constructed by piecewise multilinear interpolation:

$$S_x = \sum_{k \in C_x} v_k(x)\, S_{k^*}.$$

Here C_x denotes the set of all neighbouring integer points to x, $v_k(x)$ stands for the volume of the box with k and x as diagonally opposite vertices and with faces parallel to the coordinate planes, and k^* means the vertex opposite to k in the cube C_x.

So, we redefine $\mathcal{M}_t$ as continuous sets: $\mathcal{M}_t = \{x \in \mathbb{R}_+^d\colon S_x \geq t\}$.

Set-inclusion SLLN and LIL for renewal sets

We will need the following generalization of regular variation.

Definition 1 (Avacumović, 1936)

A non-negative measurable function p on (a, ∞), $a > 0$, is said to be $\mathcal{O}$-regularly varying ($\mathcal{O}$-RV for short) if

$$\limsup_{t \to \infty} \frac{p(ct)}{p(t)} < \infty$$

for all $c > 0$.

The class of $\mathcal{O}$-RV functions clearly includes all the RV functions together with a lot of oscillating ones.

Theorem 1 (multidimensional inversion)

Let $p = \big(p(t), t > a\big)$ *be an* $\mathcal{O}$*-RV function such that*

i) $p(t)$ *increases for large* t,

ii) $\frac{p(t)}{t}$ *decreases for large* t.

Assume that $S_n - \mu|n| = \mathcal{o}\big(p(|n|)\big)$ *a.s. as* $n \to \infty$.

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Assume that $S_n - \mu|n| = o\big(p(|n|)\big)$ *a.s. as* $n \to \infty$.

Then the inclusions $\mathcal{H}\left(\frac{\varepsilon p(t)}{t}\right) \subset t^{-1/d}\mathcal{M}_t \subset \mathcal{H}\left(-\frac{\varepsilon p(t)}{t}\right)$ *hold true a.s. for all* $\varepsilon > 0$ *and* $t > t_0$ *with some* $t_0 = t_0(\omega, \varepsilon) > 0$.

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Examples of p:

- t^r, $0 \le r \le 1$;
- $t^r(\log t)^\alpha$, $0 < r < 1$, $\alpha \in \mathbb{R}$;
- $t^r(\log t)^\alpha(\log\log t)^\delta$, $0 < r < 1$, $\alpha, \delta \in \mathbb{R}$;
- etc.

Corollary 1 (set-inclusion Marcinkiewicz-Zygmund SLLN)

Let

$$\mathbb{E}\big(|\xi|^{\beta}\log_{+}^{d-1}|\xi|\big) < \infty$$

for some $\beta \in [1, 2)$.

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Proof

Multi-indexed Marcinkiewicz-Zygmund SLLN by A. Gut (1978) + multidimensional inversion (Theorem 1).

Theorem 2 (set-inclusion LIL)

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Then

i) if $\gamma < -\mu^{-\frac{3}{2}}$, *then* $t^{-\frac{1}{d}} \mathcal{M}_t \subset \mathcal{H}\big(\gamma\sigma\sqrt{2d t^{-1} \log\log t}\big)$ *a.s. for all* $t > t_0$ *with some* $t_0 = t_0(\omega, \gamma) > 0$;

ii) if $-\mu^{-\frac{3}{2}} \le \gamma \le \mu^{-\frac{3}{2}}$, *then there are positive sequences* $\left(t_i', i \in \mathbb{N}\right)$ *and* $\left(t_i'', i \in \mathbb{N}\right)$ *depending on* ω *and* γ, *such that a.s.* $t_i' \to \infty$, $t_i'' \to \infty$, *and for all* i *a.s.*

$$(t_i')^{-\frac{1}{d}} \mathcal{M}_{t_i'} \not\subset \mathcal{H}\left(\gamma\sigma\sqrt{2d(t_i')^{-1} \log\log t_i'}\right),$$

$$(t_i'')^{-\frac{1}{d}} \mathcal{M}_{t_i''} \not\supset \mathcal{H}\left(\gamma\sigma\sqrt{2d(t_i'')^{-1} \log\log t_i''}\right);$$

iii) if $\gamma > \mu^{-\frac{3}{2}}$, *then* $t^{-\frac{1}{d}} \mathcal{M}_t \supset \mathcal{H}\big(\gamma\sigma\sqrt{2d t^{-1} \log\log t}\big)$ *a.s. for all* $t > t_0$ *with some* $t_0 = t_0(\omega, \gamma) > 0$.

Metrical SLLN and LIL for renewal sets

Definition 2

i) The Hausdorff distance $\rho_H(X, Y)$ *between two subsets* X *and* Y *of* $\mathbb{R}^d_+$ *is defined by*

$$\rho_H(X, Y) = \max\Big\{\sup_{x\in X}\inf_{y\in Y}\rho(x, y), \sup_{y\in Y}\inf_{x\in X}\rho(x, y)\Big\},$$

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ii) *For a fixed compact set* $K \subset \mathbb{R}^d$*, the localized symmetric difference distance (a.k.a. Fréchet-Nikodym one)* $\rho_\triangle^K(X, Y)$ *between two Borel subsets* X *and* Y *of* $\mathbb{R}_+^d$ *is defined by*

$$\rho_\triangle^K(X, Y) = \lambda_d\Big(K \cap \big(X \triangle Y\big)\Big),$$

with λ_d *denoting the Lebesgue measure on* $\mathbb{R}^d$*.*

Theorem 3 (metrical Marcinkiewicz-Zygmund SLLN)

Let

$$\mathbb{E}\big(|\xi|^\beta \log_+^{d-1} |\xi|\big) < \infty$$

for some $\beta \in [1,2)$.

Then

$$\rho_H\big(t^{-1/d}\mathcal{M}_t, \mathcal{H}\big) = \mathcal{O}\big(t^{\frac{1}{\beta}-1}\big) \quad \text{a.s. as } t \to \infty,$$

and, for any compact set $K \subset \mathbb{R}^d$,

$$\rho_\triangle^K\big(t^{-1/d}\mathcal{M}_t, \mathcal{H}\big) = \mathcal{O}\big(t^{\frac{1}{\beta}-1}\big) \quad \text{a.s. as } t \to \infty.$$

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In case ξ is a.s. non-negative, the bound can be improved to

$$\limsup_{t\to\infty} \frac{\rho_\triangle^K(t^{-1/d}\mathcal{M}_t, \mathcal{H})}{\sqrt{t^{-1}\log\log t}} \le \sqrt{2}\, \sigma \mu^{-\frac{3}{2}} L_K \quad a.s.$$

Thank you

for your attention