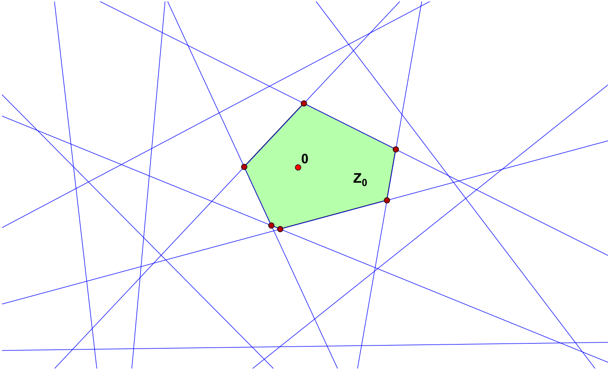
On D.G. Kendall's problem in spherical space

Daniel Hug, joint work with Andreas Reichenbacher | Luminy, May 2017

19TH CONFERENCE ON STOCHASTIC GEOMETRY, STEREOLOGY AND I

Introduction: Kendall's problem

- Poisson line process in $\mathbb{R}^2$, stationary and isotropic
- Stationary, isotropic line tessellation
- **Crofton cell or zero cell** Z_0: containing the origin



Kendall's Conjecture (1940s, 1987)

David George Kendall (1918 - 2007):

"The conditional law for the shape of Z_0, given the area $A(Z_0)$ of Z_0, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the spherical shape."

- R. Miles (1995)
- I. N. Kovalenko (1997, 1999)
- A. Goldman (1998)
- Calka (2002; '10, '13 (surveys))
- D. Hug, M. Reitzner, R. Schneider (2004)
- D. Hug, R. Schneider (2007)
- ...
- G. Bonnet (2016)
- ...

Random tessellations in $\mathbb{R}^d$

Let X be a stationary and isotropic Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\gamma > 0$. The **intensity measure** of X is

$$\mathbb{E}X(\cdot) = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{u^\perp + tu \in \cdot\}\, dt\, \sigma_{d-1}(du).$$

Let $\mathcal{H}_K := \{H : H \cap K \neq \emptyset\}$. The **hitting functional** of X is

$$K \mapsto \mathbb{E}X(\mathcal{H}_K) \sim V_1(K) \qquad \text{for } K \in \mathcal{K}^d,$$

V_1 is the **mean width**.

Let Z_0 be the zero cell of the tessellation induced by X.

What is the limit shape of Z_0 – if it exists – given $V_d(Z_0) \to \infty$?

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Kendall's problem in $\mathbb{R}^d$: a deviation result

Needed: a **deviation functional**

$\vartheta(Z_0) =$ "scaling, translation, rotation invariant distance of Z_0 from B^d".

Theorem (Hug, Reitzner, Schneider (2004), a special case **...)**

If X is stationary and isotropic in $\mathbb{R}^d$, $\varepsilon \in (0,1)$, and $a^{1/d}\gamma \geq 1$, then

$$\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon \mid V_d(Z_0) \geq a\right) \leq c \exp\left(-c_1\, \varepsilon^{d+1} a^{1/d} \gamma\right),$$

where $c = c(d, \varepsilon)$ and $c_1 = c_1(d)$.

Extensions (with Rolf Schneider): no isotropy assumption, relaxed stationarity assumption, typical cells, Voronoi and Delaunay tessellations, lower-dimensional weighted typical faces, various other size functionals, axiomatic approach, asymptotic distributions

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Kendall's problem in $\mathbb{R}^d$: asymptotic distribution

Recall: $V_1(K)$ denotes the mean width of K.

Theorem (Hug, Schneider (2007))

$$\lim_{a\to\infty} a^{-1/d} \ln \mathbb{P}(V_d(Z_0) \geq a) = -\tau\,\gamma,$$

where

$$\tau \sim \min\{V_1(K) : V_d(K) = 1\}.$$

Isoperimetric and stability problems!

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Isoperimetry and stability

Urysohn inequality:

$$V_1(K) \geq c(d)\, V_d(K)^{1/d}.$$

Equality holds if and only if K is a ball.

Quantitative stability improvement:

$$V_1(K) \geq \left(1 + a(d)\, \vartheta(K)^{d+1}\right) c(d)\, V_d(K)^{1/d}.$$

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- Large cells?
- A geometric inequality
- Some results

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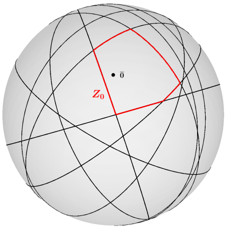
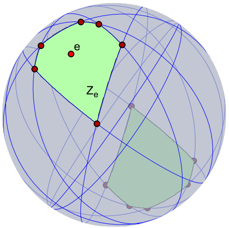
Spherical tessellations by great subspheres

- X isotropic Poisson process in $\mathbb{S}^d \subset \mathbb{R}^{d+1}$
- Spherical isotropic Poisson process of great subspheres

$$\widetilde{X} := \{x^\perp \cap \mathbb{S}^d : \; x \in X\}$$

- Crofton cell Z_0





Intensity measure and hitting functional

- Spherically convex bodies: $\mathcal{K}_s^d \ni K$
-

$$\mathcal{H}_K := \{L \in G(d+1,d) \cap \mathbb{S}^d : L \cap K \neq \emptyset\}$$

$$\mathbb{E}\widetilde{X}(\mathcal{H}_K) = \gamma_S \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap K \neq \emptyset\}\, \sigma_d(dx)$$

$$U_1(K) := (2\omega_{d+1})^{-1} \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap K \neq \emptyset\}\, \sigma_d(dx)$$

- Void probability

$$\mathbb{P}(\widetilde{X}(\mathcal{H}_K) = 0) = \exp\left(-2\gamma_S \omega_{d+1} U_1(K)\right)$$

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A spherical Urysohn inequality

Theorem (Gao, Hug, Schneider (2003))

Let $K \in \mathcal{K}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\sigma_d(C) = \sigma_d(K)$. Then

$$U_1(K) \geq U_1(C).$$

Equality holds if and only if K is a spherical cap.

Since $U_1(K) = \frac{1}{2} - V_n(K^*)$,

$$\sigma_d(C) = \sigma_d(K) \Longrightarrow \sigma_n(K^*) \leq \sigma_n(C^*),$$

and conversely.

We need a quantitative improvement / stability result!

Is K close to C (in a quantitative way), if $U_1(K)$ is ε-close to $U(C)$?

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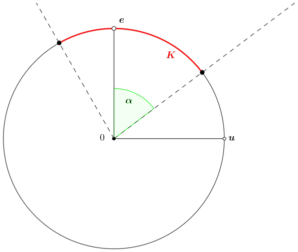

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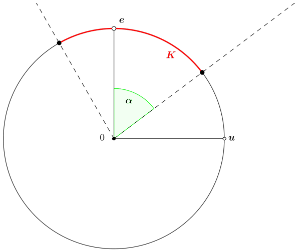

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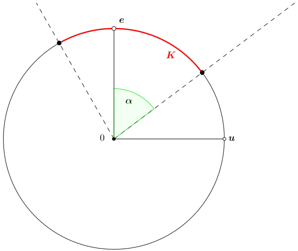

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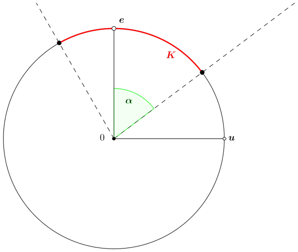

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Theorem (Hug, Reichenbacher)

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A deviation result for the spherical Crofton cell

Theorem (Hug, Reichenbacher)

Let $0 < a < \omega_{d+1}/2$ *and* $0 < \varepsilon < 1$. *Then there are constants* $\widetilde{c}_1, \widetilde{c}_2 > 0$ *such that*

$$\mathbb{P}(\Delta(Z_0) \geq \varepsilon \mid \sigma_d(Z_0) \geq a) \leq \widetilde{c}_1 \cdot \exp\left(-\widetilde{c}_2 \cdot \varepsilon^{2(d+1)} \cdot \gamma_S \cdot 2\omega_{d+1} U_1(B_a)\right),$$

where $\widetilde{c}_1 = \widetilde{c}_1(a, \varepsilon, d)$, $\widetilde{c}_2 = \widetilde{c}_2(a, d)$, B_a *is a spherical cap of volume a.*

Asymptotic distribution

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Let $0 < a < \omega_{d+1}/2$. *Then*

$$\lim_{\gamma_S \to \infty} \gamma_S^{-1} \cdot \ln \mathbb{P}(\sigma_d(Z_0) \geq a) = -2\omega_{d+1} \cdot U_1(B_a),$$

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Similar results have been obtained for binomial processes and for the spherical inradius as the size functional.

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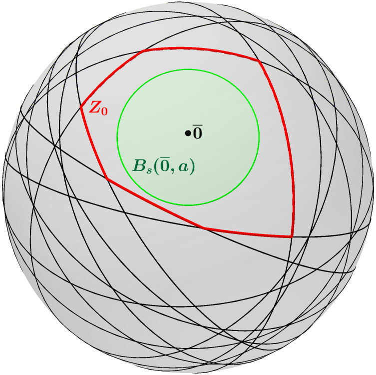
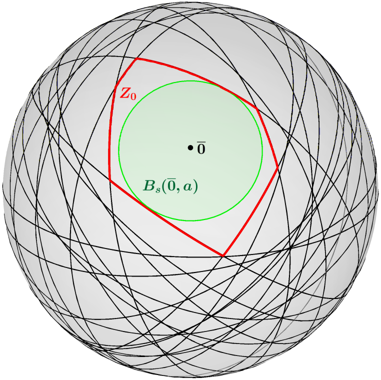


$\gamma_S = 1$ (17 great subspheres)

Illustration



$\gamma_S = 2$ (31 great subspheres)

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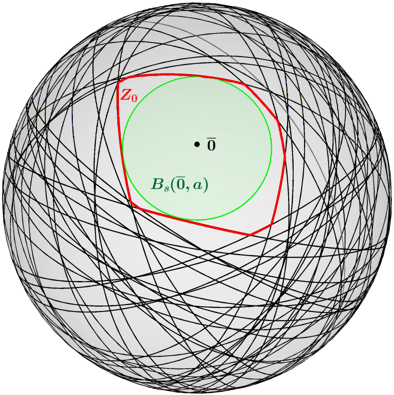

$\gamma_S = 4$ (61 great subspheres)

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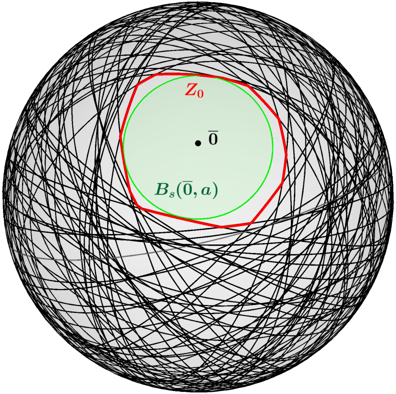

$\gamma_S = 10$ (118 great subspheres)

Typical cell

With a given isotropic tessellation X' of $\mathbb{S}^d$ with intensity $\gamma_{X'}$, we can associate particular spherical random polytopes. For a fixed point (spherical origin) $\bar{o} \in \mathbb{S}^d$, one of these is the Crofton cell $Z_0 \ni \bar{o}$.

The **typical cell** Z is a spherical random polytope, centred at $\bar{o}$, with distribution

$$\mathbb{P}(Z \in \cdot) = \frac{1}{\gamma_{X'}\omega_{d+1}} \mathbb{E}\left[\sum_{K \in X'} \int_{SO_{d+1}} \mathbf{1}\{\sigma^{-1}K \in \cdot\}\kappa(c_s(K), d\sigma)\right].$$

It is invariant wrt rotations fixing $\bar{o}$.

Crofton cell and typical cell

Lemma

Let $f : \mathcal{K}_s^d \to [0, \infty)$ be measurable and rotation invariant. Let X' be an isotropic tessellation of $\mathbb{S}^d$ with intensity $\gamma_{X'} > 0$, Crofton cell Z_0 and typical cell Z. Then

$$\mathbb{E}[f(Z_0)] = \gamma_{X'}\mathbb{E}[f(Z) \cdot \sigma_d(Z)].$$

If X' is the tessellation induced by a Poisson point process X with intensity γ_s, then $\gamma_{X'}$ is an explicitly known function of γ_s.

Typical cells of tessellations by great subspheres

The preceding Lemma and the deviation result for the Crofton cell can be combined to give a result for the typical cell.

Theorem (Hug, Reichenbacher)

Let $0 < a < \omega_{d+1}/2$ and $\varepsilon \in (0, 1]$. Let Z be the typical cell of an isotropic spherical Poisson tessellation of great subspheres. Then

$$\mathbb{P}(\Delta(Z) \geq \varepsilon \mid \sigma_d(Z) \geq a) \leq c_3 \cdot \exp\left(-c_4 \cdot \varepsilon^{2(d+1)} \cdot \gamma_S\right),$$

where $c_3 = c_3(a, d, \varepsilon)$ and $c_4 = c_4(a, d)$.

Spherical Poisson–Voronoi cells

Let X be an isotropic Poisson process on $\mathbb{S}^d$ with intensity γ_s, and let $X' = \{C(x, X) : x \in X\}$ be the associated Poisson–Voronoi tessellation.

The distribution of the typical cell Z then satisfies

$$\mathbb{P}(Z \in \cdot) = \mathbb{P}(C(\bar{o}, X + \delta_{\bar{o}}) \in \cdot).$$

Hitting and deviation functional

Hence Z is equal in distribution to the Crofton cell of a (non-isotropic) Poisson process Y of great subspheres with hitting functional

$$\mathbb{E}Y(\mathcal{H}_K) = \gamma_s \widetilde{U}(K), \qquad \bar{o} \in K \in \mathcal{K}_s^d,$$

where

$$\widetilde{U}(K) = 2 \int\limits_{\bar{o}^\perp \cap \mathbb{S}^d} \int\limits_{A_s(u)} \sin^{d-1}\left(2d_s(\tilde{S}_u, t)\right) \mathbf{1}\{t^\perp \cap K \neq \emptyset\}\, \sigma_1(dt)\, \sigma_{d-1}(du)$$

with $\tilde{S}_u = \{-\bar{o}, u\}$ and $A_s(u) = \mathrm{arc}(-\bar{o}, u)$.

Define

$$\begin{aligned} r_s(K) &:= \max\{r \geq 0 : B_s(\bar{o}, r) \subset K\} \\ R_s(K) &:= \min\{r \geq 0 : B_s(\bar{o}, r) \supset K\} \\ \vartheta(K) &:= R_s(K) - r_s(K). \end{aligned}$$

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Geometric stability

Theorem (Hug, Reichenbacher)

Let $a \in (0, \pi/2)$, $\bar{o} \in K \in \mathcal{K}_s^d$ with $r_s(K) \geq a$ and $C := B_s(\bar{o}, a)$. Then

$$\widetilde{U}(K) \geq \widetilde{U}(C) = \sigma_d(B_s(\bar{o}, 2a)).$$

Equality holds if and only if $K = C$.

More generally,

$$\widetilde{U}(K) \geq \left(1 + c_5(a, d)\, \vartheta(K)^d\right) \widetilde{U}(C).$$

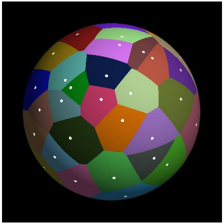
Shape deviation

Theorem (Hug, Reichenbacher)

Let $a \in (0, \pi/2)$ and $\varepsilon \in (0, 1]$. Let Z be the typical cell of the Voronoi tessellation associated with an isotropic Poisson point process with intensity γ_s on $\mathbb{S}^d$. Then

$$\mathbb{P}(R_s(Z) - r_s(Z) \geq \varepsilon \mid r_s(Z) \geq a) \leq c_6 \cdot \exp\left(-c_7 \cdot \varepsilon^d \cdot \gamma_S\right),$$

where $c_6 = c_6(a, d, \varepsilon)$ and $c_7 = c_7(a, d)$.



Davies, J. https://www.jasondavies.com/maps/voronoi