Disagreement percolation for marked Gibbs point processes

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Topic

Result

Sufficient condition for uniqueness of the Gibbs state of a Gibbs specification of a marked point process in the high temperature regime.

Using percolation, coupling and dependent thinnings.

Notation

We consider $\mathbb{R}^+$-marked configurations ω on $\mathbb{R}^d$. Marked points are $X := (x, r)$. The ball of radius r around x is $S(x, r)$ or $S(X)$.

Let Δ be a Borel set of $\mathbb{R}^d \times \mathbb{R}^+$ with bounded support in $\mathbb{R}^d$ and Ω_Δ be the locally finite marked configurations in Δ.

The Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$.

Percolation

Let $\mathcal{P}^{\text{poi}}_{\alpha,\mathcal{Q}}$ be the homogeneous marked Poisson PP with intensity α and radius (mark) measure $\mathcal{Q}$.

Gilbert graph $G(\omega)$ on ω: $(x, r) \sim (y, r')$, if $S(x, r) \cap S(y, r') \neq \emptyset$.

The Boolean model $\mathcal{P}^{\text{poi}}_{\alpha,\mathcal{Q}}$ percolates, iff

$$\mathcal{P}^{\text{poi}}_{\alpha,\mathcal{Q}}(G(\xi) \text{ contains infinite connected component}) = 1\,.$$

Percolation threshold at $\alpha(\mathcal{Q}, d) \in [0, \infty]$.

Gibbs point process

For activity $\lambda \in \mathbb{R}^+$, radius measure $\mathcal{Q}$, domain Δ and boundary condition $\tilde{\omega} \in \Omega_{\Delta^c}$:

$$\mathcal{P}^{gibbs}_{\Delta,\tilde{\omega},\lambda,\mathcal{Q}}(\,\mathrm{d}\omega) := \frac{\lambda^{|\omega|}\exp(-H_\Delta(\omega|\tilde{\omega}))(\mathcal{L}^d \otimes \mathcal{Q})^{|\omega|}(\mathrm{d}\omega)}{Z(\Delta,\lambda,\mathcal{Q},\tilde{\omega})},$$

with the partition function $Z(\Delta,\lambda,\mathcal{Q},\tilde{\omega})$. Fulfils DLR, assume existence of Gibbs states.

Stochastic domination

$\mathcal{P}^1$ stochastically dominates $\mathcal{P}^2$, iff there exists a coupling $\mathcal{P}$ of them with $\mathcal{P}(\xi^1 \geq \xi^2)$. "More and bigger points."

Sufficient condition for stochastic domination: Papangelou intensity $\rho^1(X,\omega) \geq \rho^2(X,\omega)$. Preston 76,Georgii & Küneth 97

Necessary properties

Locality

The interaction occurs within connected components of the Gilbert graph. $H(\omega|\tilde{\omega})$ depends only on the connected components of $G(\omega \cup \tilde{\omega})$ intersecting ω.

Boundedness

The Papangelou intensity is uniformly bounded

$$\lambda \exp(-H(X|\tilde{\omega})) \leq \alpha\,.$$

Models

Finite range repulsive interaction, continuum random cluster, Widom-Rowlinson, quermass-interaction.

Disagreement coupling

Coupling of 3 point processes (van den Berg & Maes 94)

Suppose that, for all $\Delta \subseteq S$ with $\operatorname{supp}\Delta \Subset \mathbb{R}^d$, $\tilde{\omega}_1, \tilde{\omega}_2 \in \Omega_{\Delta^c}$, there exists $\mathcal{P} := \mathcal{P}_{\Delta,\tilde{\omega}_1,\tilde{\omega}_2}$ with

$$i \in \{1,2\}: \quad \mathcal{P}(\xi^i = \mathrm{d}\omega) = \mathcal{P}^{gibbs}_{\Delta,\tilde{\omega}_i,\lambda,\mathcal{Q}}(\mathrm{d}\omega)$$

$$\mathcal{P}(\xi^3 = \mathrm{d}\omega) = \mathcal{P}^{\mathsf{poi}}_{\Delta,\alpha,\mathcal{Q}}(\mathrm{d}\omega)$$

$$\mathcal{P}(\xi^1 \triangle \xi^2 \leq \xi^3) = 1$$

$$\mathcal{P}(\forall X \in \xi^1 \triangle \xi^2 : X \xleftrightarrow{\text{in } G(\xi^3)} \tilde{\omega}_1 \triangle \tilde{\omega}_2)\,.$$

Disagreement percolation

Theorem

If $\mathcal{P}^{poi}_{\alpha,\mathcal{Q}}$ does not percolate ($\alpha < \alpha(\mathcal{Q}, d)$), then there is a unique Gibbs state.

Theorem

If the connection function of $\mathcal{P}^{poi}_{\alpha,\mathcal{Q}}$ decays exponentially, then the pair correlation of the Gibbs states decays exponentially, too.

Dependent thinning

Couple $\mathcal{P}^{gibbs}_{\Delta,\tilde{\omega},\lambda,\mathcal{Q}}$ and $\mathcal{P}^{\mathrm{poi}}_{\Delta,\alpha,\mathcal{Q}}$ by a dependent thinning. Explore ω drawn from $\mathcal{P}^{\mathrm{poi}}_{\Delta,\alpha,\mathcal{Q}}$ in (measurable total) order. At $X \in \omega$: having chosen $\gamma \subseteq \omega \cap]-\infty, X[$, choose X with probability

$$
\begin{aligned}
p_\Delta(X|\gamma,\tilde{\omega}) &:= \frac{1}{\alpha}\frac{\partial}{\partial X} \log Z([X,\infty[,\lambda,\mathcal{Q},\gamma\cup\tilde{\omega}) \\
&= \underbrace{\frac{\lambda \exp(-H(X|\gamma\cup\tilde{\omega}))}{\alpha}}_{\leq 1} \underbrace{\frac{Z(]X,\infty[,\lambda,\mathcal{Q},\gamma\cup\tilde{\omega}\cup X)}{Z([X,\infty[,\lambda,\mathcal{Q},\gamma\cup\tilde{\omega})}}_{\leq 1} .
\end{aligned}
$$

Reduces to Papangelou intensity in extreme case.

Summary & outlook

Comparison with other uniqueness criteria Cluster expansion Ruelle 69,Hofer-Temmel 15-17+ Dobrushin uniqueness Klein 82

Models

Applications Uniqueness, Poincare inequality for dynamics Chazottes & Redig & Völlering 11.

Generalisations

- Replace $\mathbb{R}^+$ marks by $\mathbb{R}^k$ (easy) or compact bodies (difficult?).
- Stochastic domination also in $\mathcal{Q}$, i.e., $\mathcal{Q} \preccurlyeq \mathcal{Q}'$. Finer constraint than uniformly bounded Papangelou intensity.
- Factorisation of joint thinning probability over connected components of $G(\omega \cup \tilde{\omega})$.

Bibliography

Disagreement percolation in the study of Markov fields van den Berg & Maes 94

Disagreement percolation for the hard-sphere model Hofer-Temmel 15-17+

The Poincaré inequality for Markov random fields proved via disagreement percolation Chazottes & Redig & Völlering 11

Continuum percolation Meester & Roy 96

Random Fields Preston 76

Stochastic comparison of point random fields Georgii & Küneth 97

Dobrushin uniqueness techniques and the decay of correlations in continuum statistical mechanics Klein 82

Statistical mechanics Ruelle 69