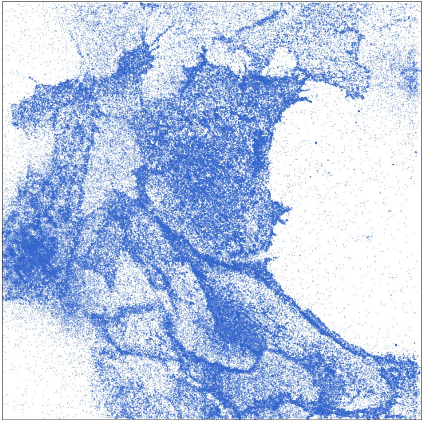
Cluster marked cluster point processes

Ute Hahn

CENTRE FOR **STOCHASTIC GEOMETRY** AND ADVANCED **BIOIMAGING**

Aarhus University,
Denmark

19th Workshop on
Stochastic Geometry, Stereology and Image Analysis
Luminy 15.05.2017



Positions of aquaporin molecules on kidney cell membrane

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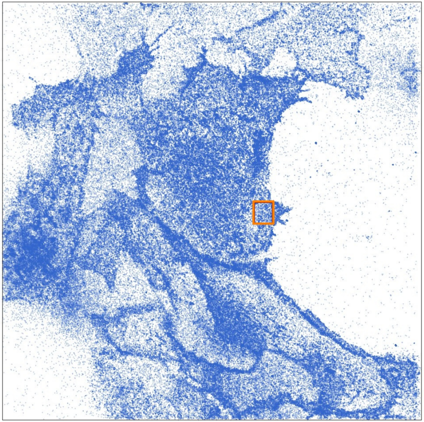

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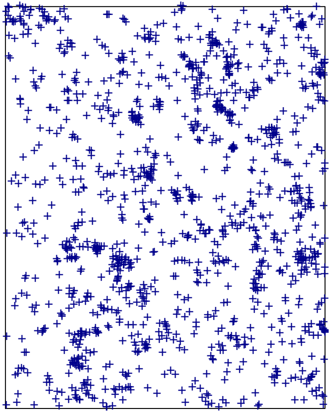
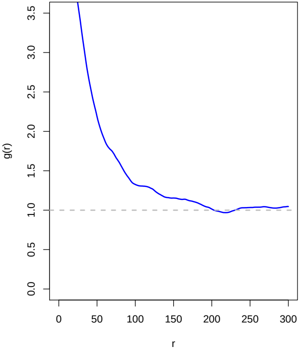
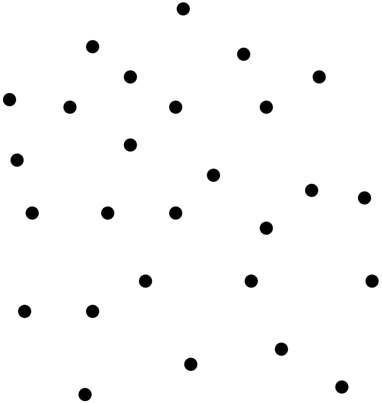


ca. 3300 × 4100 nm

Aim: characterize clustering by pair correlation function



$$g(r) = \frac{\text{Palm intensity of } \boldsymbol{X} \text{ at distance } r \text{ to typical point}}{\text{Intensity of } \boldsymbol{X}}$$



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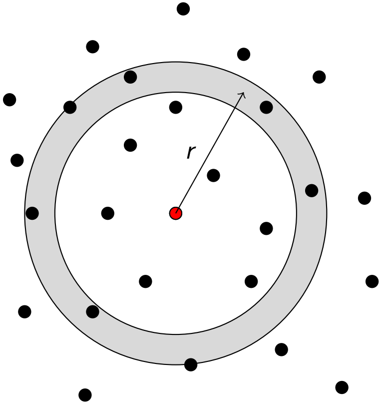

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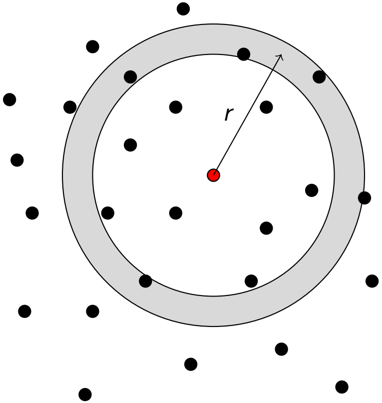
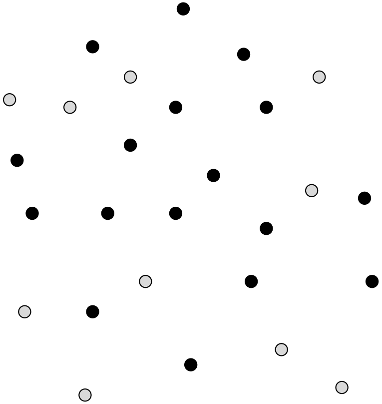


The pair correlation function is invariant to independent thinning of the process.



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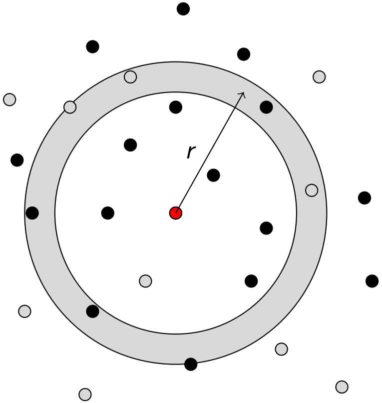

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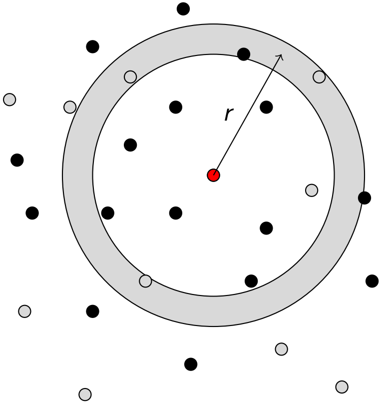

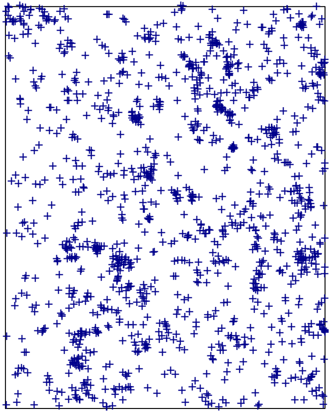

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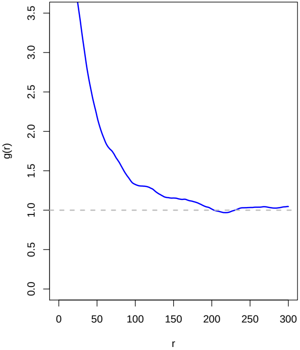
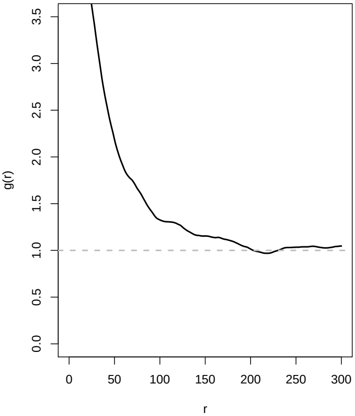


original pair correlation of $\boldsymbol{X}$:

`pcf(X)`

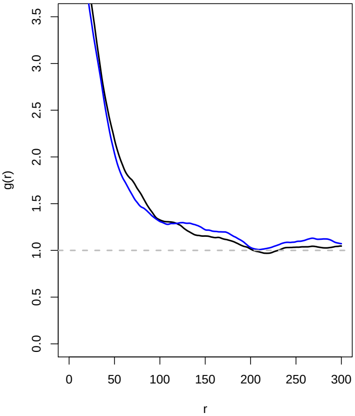


original pair correlation of $\boldsymbol{X}$:

```
pcf(X)
```

indep.thinning, retain 500 points:

```
pcf(X[sample(npoints(X), 500)])
```



original pair correlation of ***X***:

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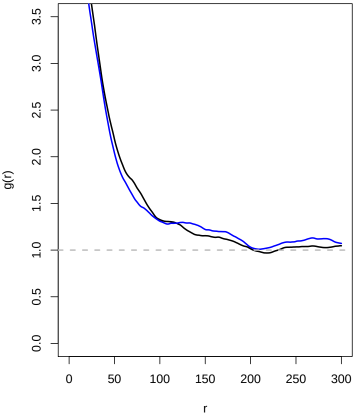
indep.thinning, retain 500 points:

```
pcf(X[sample(npoints(X), 500)])
```

first 500 points:

```
pcf(X[1:500])
```



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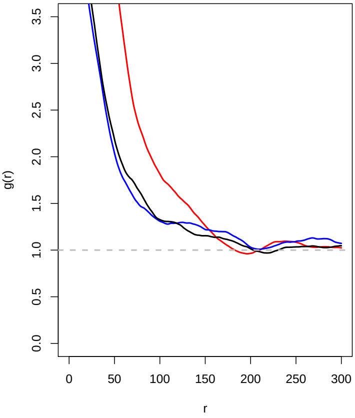


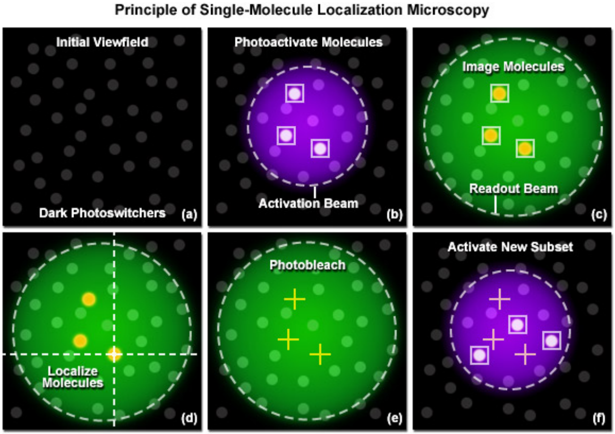
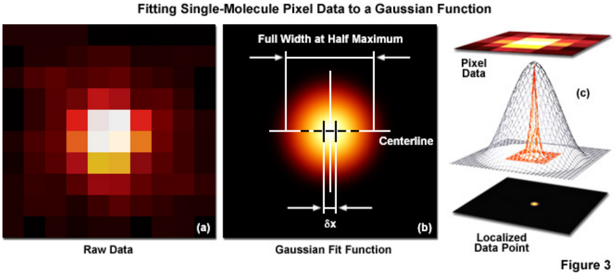


Figure 1

Published 2006, Method of the Year 2008, Nobel Prize in Chemistry 2014 (Betzig, Hell, Moerner)

Source: http://zeiss-campus.magnet.fsu.edu

Molecule positions estimated from digital images by maximum likelihood



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Scientific hypothesis on PALM images:

- Points represent positions of fluorescent molecules,
- molecules fluoresce independently of each other.

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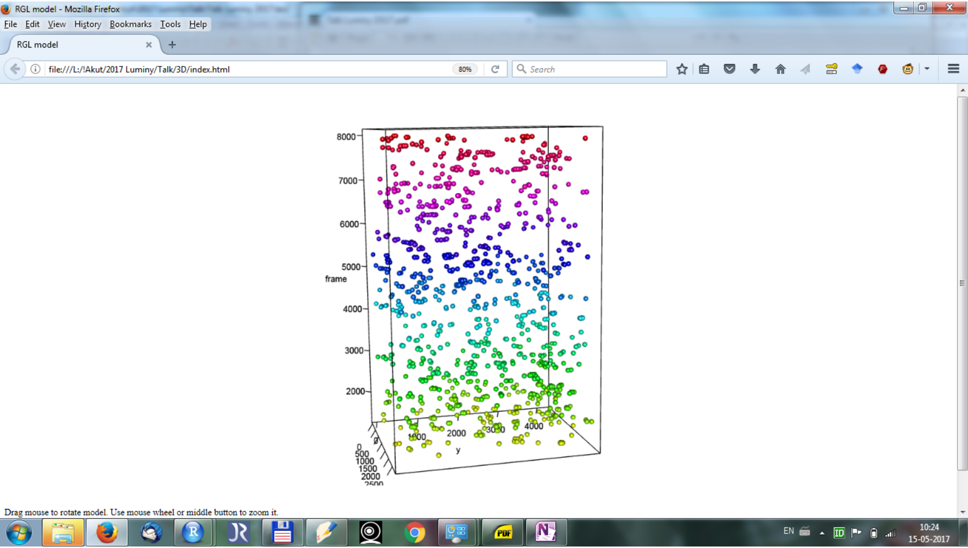
Data:

- points registered in sequence of video frames,
- frame number available (range:$[1400, 8000]$).

☞ analyse as space-time or marked point pattern;
inspect data in 3D, as points in $\mathbb{R}^2 \times$ frames.


RGL model - Mozilla Firefox
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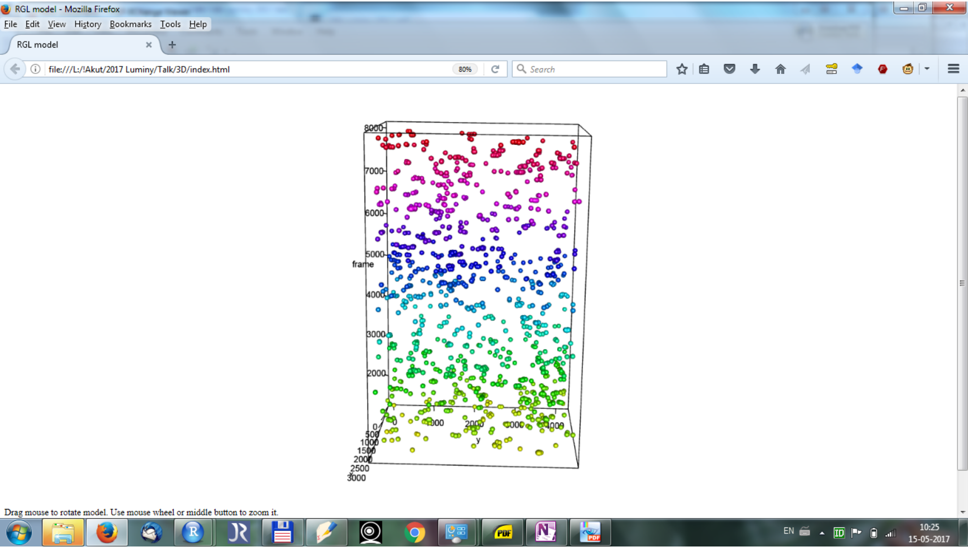
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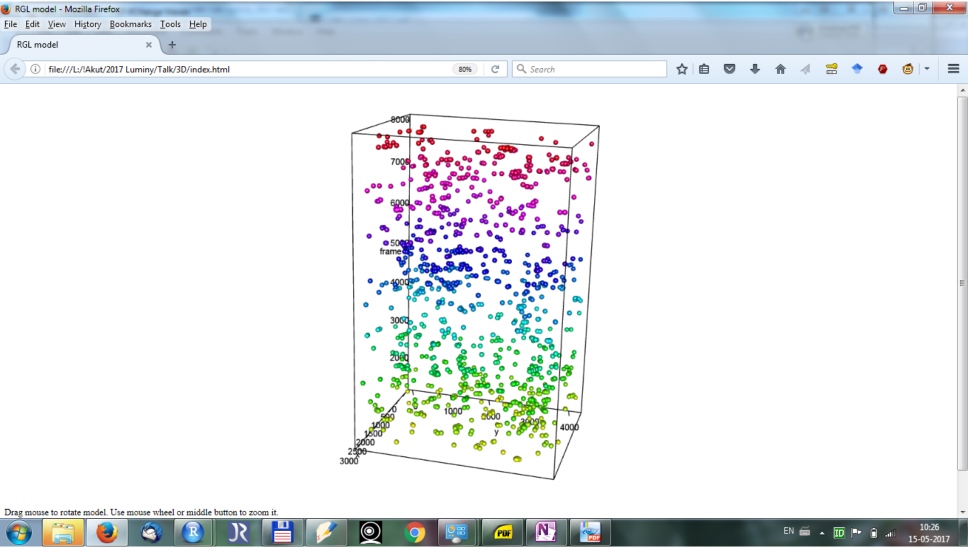

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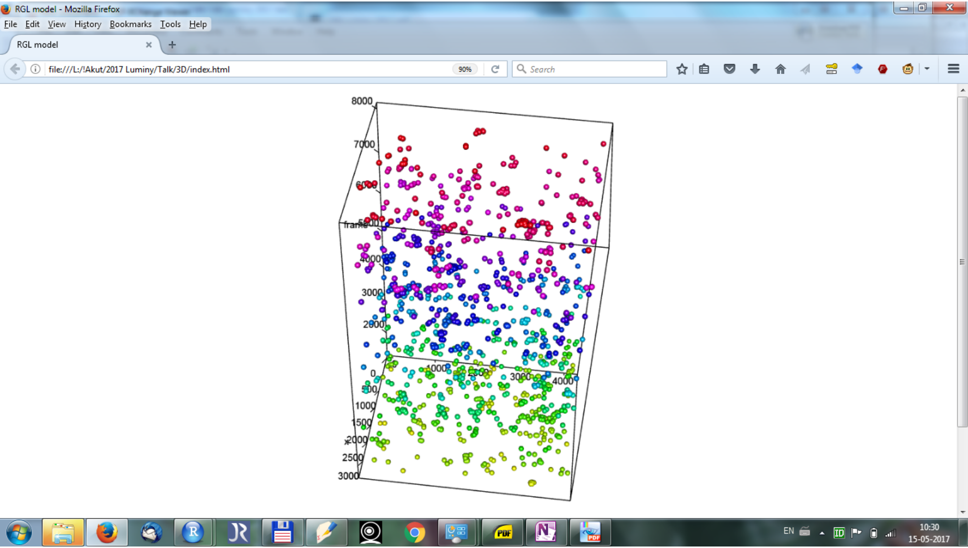
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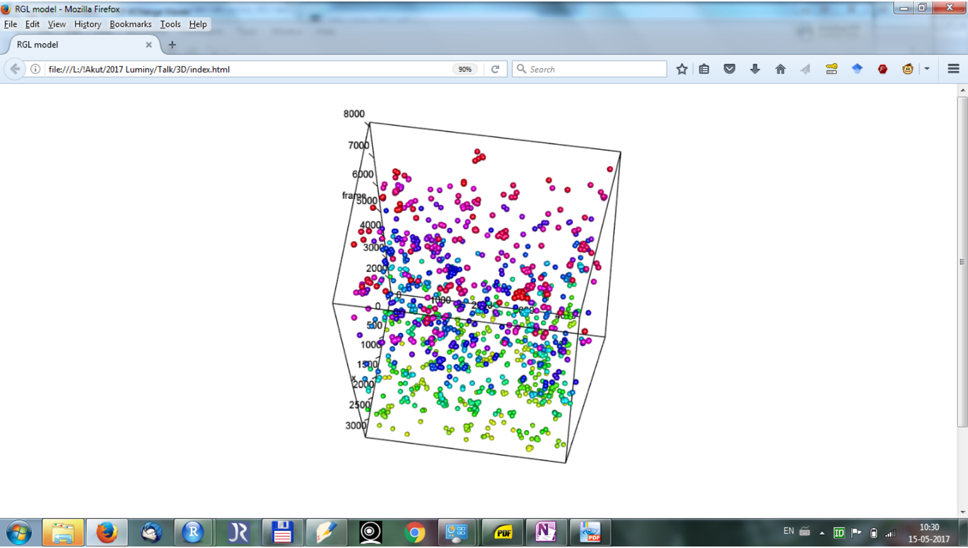
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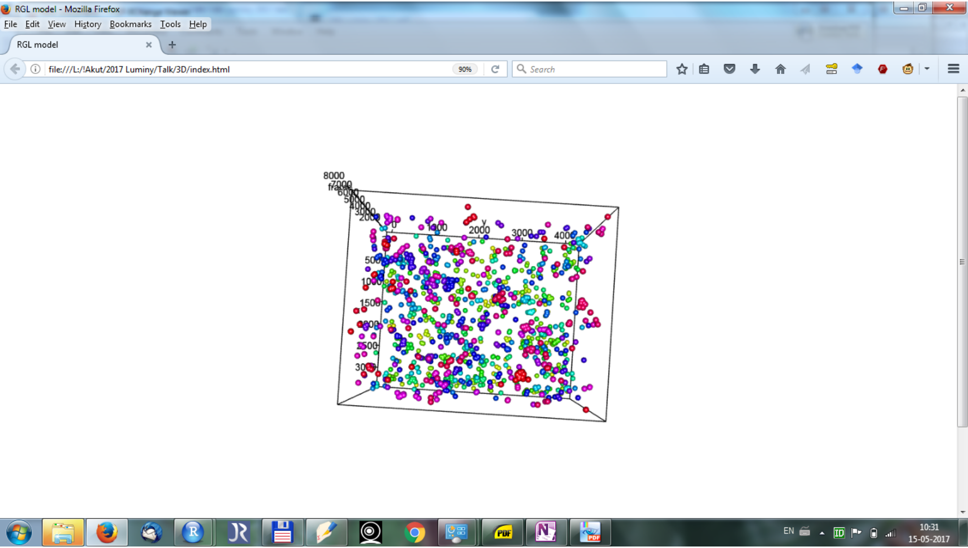
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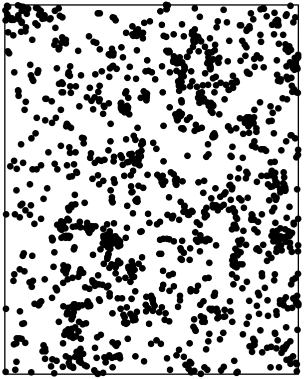
Ground process: cluster process $\boldsymbol{X}$

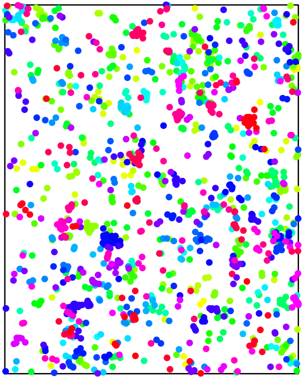
- Parent process $\boldsymbol{Y}$
- cluster template $\boldsymbol{C}_0$
- independent clusters $\boldsymbol{C}(y) \sim \boldsymbol{C}_0 + y$, for all $y \in \boldsymbol{Y}$
- superposition of clusters

$$\boldsymbol{X} = \bigcup_{y \in \boldsymbol{Y}} \boldsymbol{C}(y).$$



Parent dependent marking

- Notation: $m(u)$ mark of point u, mark space $\mathbb{M}$.
- Location family of mark distributions for the clusters: $\{\mathcal{M}_\theta, \theta \in \Theta\}$.
- Parent process is i.i.d. marked, $m(y) \sim \mathcal{T}$, with values in Θ.
- Points $u \in \boldsymbol{C}(y)$ are i.i.d. marked with $m(u) \sim \mathcal{M}_{m(y)}$.



Resulting cluster marked cluster process (CMCpp) is mark stationary: Palm mark distribution

$$\mathcal{M}(\mathrm{d}\, m) = \mathbf{E}\mathcal{M}_T(\mathrm{d}\, m), \quad T \sim \mathcal{T},$$

is independent of location.

Appropriate model for the data:

- Mark distributions $\mathcal{M}_\theta$ have small variance,
- distribution $\mathcal{T}$ of location parameter spreads widely.

☞ Points close to each other in space have preferentially similar frame marks.

Want: a statistics that shows that spatial clusters cover only a close range of marks (frames).

Stoyan's mark correlation function for stationary isotropic point process $\boldsymbol{X}$: f measureable function on $\mathbb{M} \times \mathbb{M}$

$$k_f(r) := \frac{\mathbf{E}[f(m(u), m(v)) \mid u, v \in \boldsymbol{X}]}{\mathbf{E}[f(M_1, M_2)]},$$
$$\|u - v\| = r,\ M_1, M_2 \sim \mathcal{M}, \text{indep.}$$

- If $\boldsymbol{X}$ is independently marked, $k_f(r) \equiv 1$.
- For (quasi) continuous marks, the usual choice is $f(m_1, m_2) = m_1 m_2$.

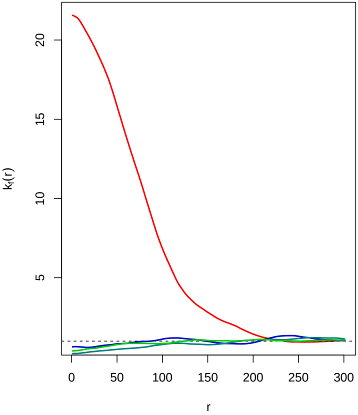
- Under the model, $|m(u) - m(v)|$ is small for points from same cluster, but likely to be large for points from different clusters.
- Choose $f(m_1, m_2) = \mathbf{1}(|m_1 - m_2| \in [t, t + \Delta])$, thus

$$k_f(r) = \frac{\Pr(|m(u) - m(v)| \in [t, t + \Delta] \mid u, v \in \boldsymbol{X}, \|u - v\| = r)}{\Pr(|m(u) - m(v)| \in [t, t + \Delta] \mid u, v \in \boldsymbol{X})}$$

– an odds ratio.

Results, $\Delta = 20$:

- $t = 0$:
  emphasize points close in time
- $t = 20$, $t = 40$, $t = 60$



interpretation: close points (distance $r < 200$), are more likely to have similar frame numbers than different frame numbers.

Theorem.

Let:

$\boldsymbol{X}$ be a cluster marked cluster process with pair correlation function $g_{\boldsymbol{X}}$,
$\boldsymbol{Y}$ its parent process,
$\tilde{\boldsymbol{C}}$ the distribution of a random point in template clusters $\boldsymbol{C}$,
$\mathcal{M}_\theta, \theta \in \Theta$ the mark distributions of the clusters,
$\mathcal{M}$ the overall Palm mark distribution.

Let f be a function on $\mathbb{M}^2$ that fulfils, for independent M_1, M_2:

$$\mathbf{E} f(M_1, M_2) \begin{cases} = 0, & M_1, M_2 \sim \mathcal{M}_\theta, \text{ for all } \theta \in \Theta, \\ > 0, & M_1, M_2 \sim \mathcal{M} \end{cases}$$

then for the mark correlation function k_f of $\boldsymbol{X}$,

$$k_f(r) g_{\boldsymbol{X}}(r) = g_{\tilde{\boldsymbol{Y}}}(r),$$

where $\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{C}} * \boldsymbol{Y}$ is the randomly $\tilde{\boldsymbol{C}}$-displaced parent process.

f-weighted second order factorial moment measure

$$\alpha_f^{(2)}(B_1, B_2) = \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

independent marking:

$$\alpha_f^{(2)}(B_1, B_2) = \underbrace{\mathbf{E} f(M_1, M_2)}_{=:\mu} \underbrace{\mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)}_{=\alpha^{(2)}(B_1, B_2)}, \quad M_1, M_2 \sim \mathcal{M}, \text{indep.}$$

☞ mark correlation function

$k_f(\,,\,)$ is the density of $\alpha_f^{(2)}$ wrt. $\mu\alpha^{(2)}$.

independent locations (Poisson process)

$$\alpha^{(2)}(B_1, B_2) = \Lambda(B_1)\Lambda(B_2) =: \Lambda^2(B1, B2) \qquad \Lambda : \text{intensity measure}$$

☞ pair correlation function

$g(\,,\,)$ is the density of $\alpha^{(2)}$ wrt. Λ^2

independent locations (Poisson process)

$$\alpha^{(2)}(B_1, B_2) = \Lambda(B_1)\Lambda(B_2) =: \Lambda^2(B1, B2) \qquad \Lambda : \text{intensity measure}$$

☞ pair correlation function

$$g(\,,\,) \text{ is the density of } \alpha^{(2)} \text{ wrt. } \Lambda^2$$

thus, with k_f being the density of $\alpha_f^{(2)}$ wrt. $\mu\alpha^{(2)}$,

$$k_f \cdot g \text{ is the density of } \alpha_f^{(2)} \text{ wrt. } \mu\,\Lambda^2.$$

Cluster marked process

$$
\begin{aligned}
\alpha_f^{(2)}(B_1, B_2) &= \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\
&= \mathbf{E} \sum_{y \in \boldsymbol{Y}} \sum_{u,v \in \boldsymbol{C}(y)}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\
&\quad + \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)
\end{aligned}
$$

Cluster marked process

$$\begin{aligned}\alpha_f^{(2)}(B_1, B_2) &= \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\ &= \mathbf{E} \sum_{y \in \boldsymbol{Y}} \sum_{u,v \in \boldsymbol{C}(y)}^{\neq} \mathbf{E} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\ &\quad + \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} \mathbf{E} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)\end{aligned}$$

Cluster marked process

$$\alpha_f^{(2)}(B_1, B_2) = \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

$$= \mathbf{E} \sum_{y \in \boldsymbol{Y}} \sum_{u,v \in \boldsymbol{C}(y)}^{\neq} \mu_{C(y)} \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

$$+ \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} \mu \, \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

Cluster marked process

$$\alpha_f^{(2)}(B_1, B_2) = \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

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$$+ \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} \mu\, \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

with f such that $\mu_{\boldsymbol{C}(y)} = 0$,

$$= \mu\ \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \tilde{\boldsymbol{C}}} \sum_{v \in \tilde{\boldsymbol{C}}'} \mathbf{1}_{B_1}(y+u) \mathbf{1}_{B_2}(y'+v)$$

Cluster marked process

$$\alpha_f^{(2)}(B_1, B_2) = \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

$$= \mathbf{E} \sum_{y \in \boldsymbol{Y}} \sum_{u,v \in \boldsymbol{C}(y)}^{\neq} \mu_{C(y)} \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

$$+ \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} \mu \, \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)$$

with f such that $\mu_{\boldsymbol{C}(y)} = 0$,

$$= \mu \; \nu^2 \mathbf{E} \sum_{y,y' \in \tilde{\boldsymbol{Y}}}^{\neq} \mathbf{1}_{B_1}(y) \mathbf{1}_{B_2}(y')$$

ν: mean number of points in $\boldsymbol{C}$

Cluster marked process

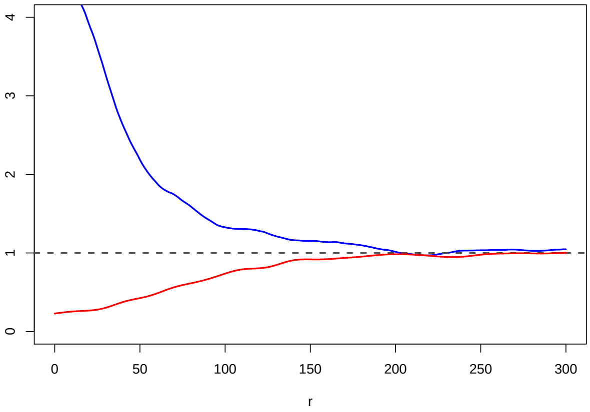
$$
\begin{aligned}
\alpha_f^{(2)}(B_1, B_2) &= \mathbf{E} \sum_{u,v \in \boldsymbol{X}}^{\neq} f(m(u), m(v)) \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\
&= \mathbf{E} \sum_{y \in \boldsymbol{Y}} \sum_{u,v \in \boldsymbol{C}(y)}^{\neq} \mu_{C(y)} \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v) \\
&\quad + \mathbf{E} \sum_{y,y' \in \boldsymbol{Y}}^{\neq} \sum_{u \in \boldsymbol{C}(y)} \sum_{v \in \boldsymbol{C}(y')} \mu \, \mathbf{1}_{B_1}(u) \mathbf{1}_{B_2}(v)
\end{aligned}
$$

with f such that $\mu_{\boldsymbol{C}(y)} = 0$,

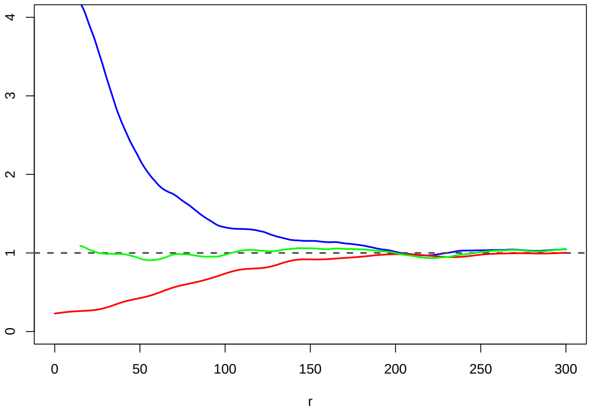
$$
\begin{aligned}
&= \mu \, \nu^2 \mathbf{E} \sum_{y,y' \in \tilde{\boldsymbol{Y}}}^{\neq} \mathbf{1}_{B_1}(y) \mathbf{1}_{B_2}(y') \\
&= \mu \nu^2 \, \alpha_{\tilde{\boldsymbol{Y}}}^{(2)}(B_1, B_2).
\end{aligned}
$$

pair correlation function $g(r)$

mark correlation function $k_f(r)$ with $f(m_1, m_2) = |m_1 - m_2|$

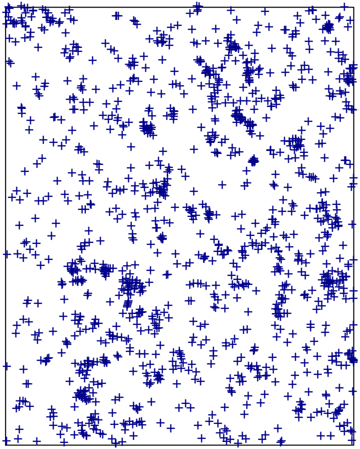


pair correlation function $g(r)$
mark correlation function $k_f(r)$ with $f(m_1, m_2) = |m_1 - m_2|$
$k_f(r)g(r)$



Model for ground process:

Matérn cluster process superimposed with Poisson "noise".



data

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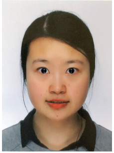

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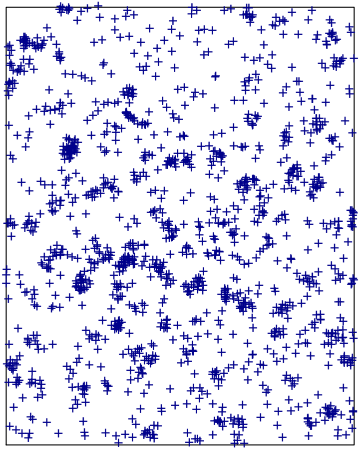

fittet model