Cluster size distribution of extreme values in a Poisson-Voronoi tessellation

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Framework

- η : Poisson point process;
- $\mathrm{m}_{PVT} := \{C_\eta(x) : x \in \eta\}$: Voronoi tessellation associated with η;
- g: geometric characteristic (e.g. volume, diameter);
- $v \geq 0$: threshold.

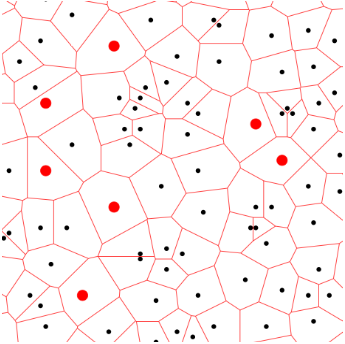


Figure: Nuclei of the Voronoi cells with diameter larger than v in a square

Plan

Notation

- η : homogeneous Poisson point process of intensity 1 in $\mathbf{R}^d$;
- g: translation-invariant function;
- $W_\rho := \rho^{1/d}[-\frac{1}{2}, \frac{1}{2}]^d$, $\rho > 0$;
- v_ρ: threshold such that

$$\rho\mathbb{P}\left(g(\mathcal{C}) > v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} \tau,$$

  for some $\tau > 0$, where $\mathcal{C} = C_{\eta\cup\{0\}}(0)$ is the typical cell;
- $\Phi^\eta_{W_\rho}$: (normalized) point process of exceedances in W_ρ, i.e.

$$\Phi^\eta_{W_\rho} := \rho^{-1/d}\{x \in \eta \cap W_\rho : g(C_\eta(x)) > v_\rho\}.$$

NB: $\tau = \lim_{\rho\to\infty} \mathbb{E}\left[\#\Phi^\eta_{W_\rho}\right]$ is (asymptotically) the mean number of exceedances in W_ρ.

Convergence of the point process of exceedances

Theorem 1 (under a local condition)

Let v_ρ be such that $\rho\mathbb{P}(g(\mathcal{C}) > v_\rho) \underset{\rho\to\infty}{\longrightarrow} \tau$. Assume that the following local condition holds:

$$\rho \int_{B(0;\log\rho)} \mathbb{P}\left(g(C_{\eta\cup\{0,x\}}(0)) > v_\rho, g(C_{\eta\cup\{0,x\}}(x)) > v_\rho\right) \mathrm{d}x \underset{\rho\to\infty}{\longrightarrow} 0.$$

Then

$$\Phi^{\eta}_{W_\rho} = \rho^{-1/d}\{x \in \eta \cap W_\rho : g(C_\eta(x)) > v_\rho\}$$

converges to a Poisson point process in $[-\frac{1}{2}, \frac{1}{2}]^d$ with intensity τ.

NB: since $\{\max_{x\in\eta\cap W_\rho} g(C_\eta(x)) \leq v_\rho\} = \{\#\Phi^{\eta}_{W_\rho} = 0\}$, we have:

$$\mathbb{P}\left(\max_{x\in\eta\cap W_\rho} g(C_\eta(x)) \leq v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau}.$$

Extremal index

Proposition

Assume that for each $\tau \geq 0$, there exists $v_\rho(\tau)$ such that $\rho\mathbb{P}(g(\mathcal{C}) > v_\rho(\tau)) \underset{\rho\to\infty}{\longrightarrow} \tau$. Then there exists $\theta \in [0,1]$ such that, for each $\tau \geq 0$,

$$\lim_{\rho\to\infty} \mathbb{P}\left(\max_{x\in\eta\cap W_\rho} g(C_\eta(x)) \leq v_\rho(\tau)\right) = e^{-\theta\tau},$$

provided that the limit exists.

Definition

According to Leadbetter, we say that $\theta \in [0,1]$ is the extremal index if, for each $\tau \geq 0$, we have:

1. $\rho\mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} \tau$;
2. $\mathbb{P}\left(\max_{x\in\eta\cap W_\rho} g(C_\eta(x)) \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}$.

Remarks

- Characterization of the extremal index:
  - discretization of $W_\rho = \rho^{1/d}[-\frac{1}{2}, \frac{1}{2}]^d$ into k_ρ^d blocks of equal size, with $k_\rho^d = o(\rho)$ and $\log \rho = o(k_\rho^d)$;
  - interpretation:

$$\theta^{-1} = \lim_{\rho\to\infty} \frac{\mathbb{E}\left[\text{Nbr of cells exceeding } v_\rho(\tau)\right]}{\mathbb{E}\left[\text{Nbr of blocks exceeding } v_\rho(\tau)\right]}.$$

- In the classical Extreme Value Theory:
  - computation of θ: difficult;
  - estimation of θ: block and run methods.

- For a Poisson-Voronoi tessellation:
  - block method: can be adapted, but not efficient;
  - run method: cannot be adapted (specific to the unidimensional case).

Cluster size distribution

- $\Phi_B^\eta(\tau) := \rho^{-1/d}\{x \in \eta \cap B : g(C_\eta(x)) > v_\rho(\tau)\}$ for any $B \subset \mathbf{R}^d$;
- $Q_\rho = q_\rho^{1/d}[-\frac{1}{2}, \frac{1}{2}]^d$, with $q_\rho = (\log\log\rho)^{\log\log\rho} \ll \rho$.

Theorem 2 (without local condition)

Assume that the limit $\lim_{\rho\to\infty} \mathbb{P}\left(\#\Phi_{W_\rho}^\eta(\tau_0) = 0\right) \in (0,1)$ exists, for some $\tau_0 > 0$, and assume that the limit distribution $p = (p_k)$ exists, with:

$$p_k := \lim_{\rho\to\infty} \mathbb{P}\left(\#\Phi_{Q_\rho}^{\eta\cup\{0\}}(\tau_0) = k \,\middle|\, g(C_{\eta\cup\{0\}}(0)) > v_\rho(\tau_0)\right), k \geq 1.$$

Then

1. $\theta = \sum_{k=1}^\infty k^{-1} p_k$;
2. $\Phi_{W_\rho}^\eta(\tau)$ converges, for each $\tau > 0$, to a compound Poisson point process in $[-\frac{1}{2}, \frac{1}{2}]^d$, with intensity $\theta\tau > 0$, and cluster size distribution (π_k) with $\pi_k := \frac{p_k}{k\theta}$.

Layout

Numerical illustrations for two geometric characteristics (dimension $d = 2$):

- approximations of the probabilities $p_1, \ldots, p_9$ and approximation of the extremal index:
$$\theta = \sum_{k=1}^{\infty} k^{-1} p_k;$$
- $\tau = 1$, $\rho = \exp(100)$, $Q_\rho = q_\rho^{1/2}[-\frac{1}{2}, \frac{1}{2}]^2 = [-173, 173]^2$;
- computation of $v_\rho(1)$ such that $\rho \mathbb{P}\left(g(\mathcal{C}) > v_\rho(1)\right) \underset{\rho\to\infty}{\longrightarrow} 1$;
- 10000 simulations divided into 100 samples of size 100, such that $g(\mathcal{C}) > v_\rho(1)$.

Example 1: inradius

- $g(C_\eta(x)) = r(C_\eta(x)) =$ inradius of $C_\eta(x)$;
- condition (LCC) holds;
- $\theta = 1$, $\pi_1 = 1$ and $\pi_k = 0$ for $k \geq 2$.

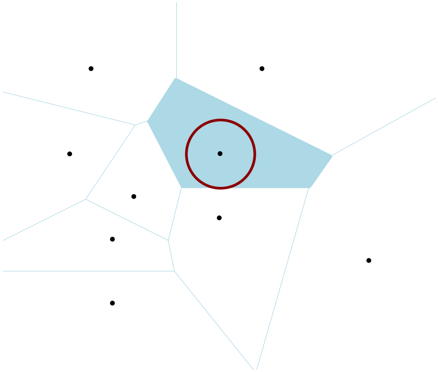
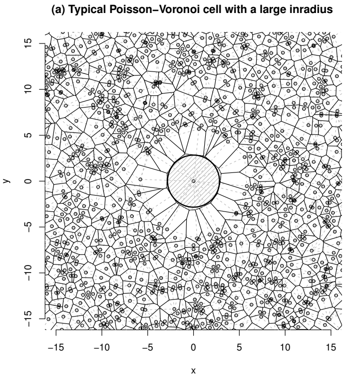


Figure: incircle (red) of a Voronoi cell (green).

Numerical illustration



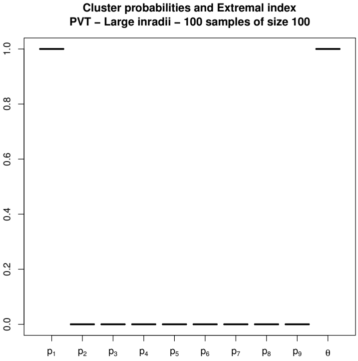
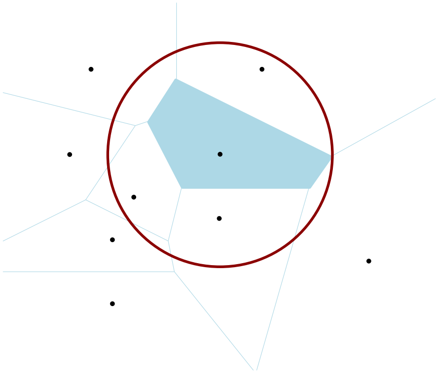


Figure: typical cell with a large inradius (left) and estimates of p_k (right).

Example 2: circumradius

- $g(C_\eta(x)) = R(C_\eta(x)) =$ circumradius of $C_\eta(x)$;
- maximum of circumradii: interpretation as a covering of the window W_ρ by balls.
- $\theta = \frac{1}{4}$? π_k?



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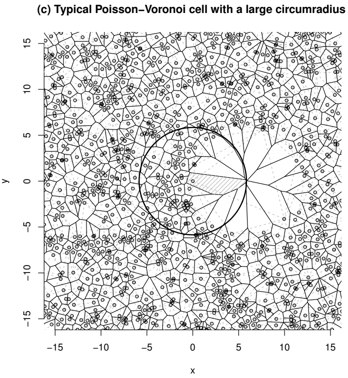

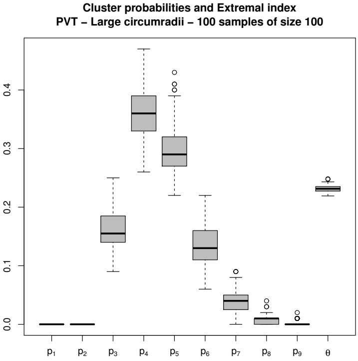

Figure: typical cell with a large circumradius (left) and estimates of p_k (right).

Perspectives

- existence of the extremal index;
- adaptation of the main theorem (without local condition) to various random tessellations;
- theoretical computations of extremal indices and of distributions (π_k) for various geometric characteristics;
- shape of a cluster of exceedances.