Voronoi diagram on a Riemannian manifold

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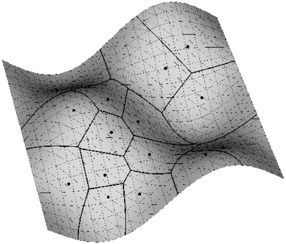
Motivation

Aim :

- Show a link between mean characteristics of the Voronoi cells and local characteristics of the manifold
- Derive limit theorems to develop statistical tools



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Framework

- M Riemannian manifold of dim n, with its Riemannian metric d,
- dx the measure induced by the metric,
- $\mathcal{P}_\lambda$ Poisson point process of intensity λdx and $x_0 \in M$ **fixed**,
- The Voronoi cell of $x \in \mathcal{P}_\lambda$ is defined by

$$C(x, \mathcal{P}_\lambda) = \{y \in M, d(x,y) \leq d(x', y), \forall x' \in \mathcal{P}_\lambda\}$$

- $\mathcal{C} = C(x_0, \mathcal{P}_\lambda \cup \{x_0\})$ is the Voronoi cell of x_0,
- $N(C(x, \mathcal{P}_\lambda))$ the number of vertices of $C(x, \mathcal{P}_\lambda)$.

Outline

1. Mean number of vertices of $\mathcal{C}$
2. Limit theorems and estimation
3. Probabilistic proof of Gauss-Bonnet theorem

Mean number of vertices of $\mathcal{C}$

Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = E_n - \frac{\mathrm{Sc}(x_0)}{\lambda^{\frac{2}{n}}} C_n + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

with

- E_n is the mean number of vertices in the case of $\mathbb{R}^n$,
- C_n is a positive constant,
- $\mathrm{Sc}(x_0)$ is the scalar curvature of M at x_0.

Remarks:

1. Mean number of vertices in a given direction $\rightsquigarrow$ Ricci curvature
2. Sectional Voronoi cell $\rightsquigarrow$ sectional curvature

Sketch of proof

Each vertex of $\mathcal{C}$ is a circumcenter of x_0 and n points of the process.

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}[\sum_{x_1,\dots,x_n \in \mathcal{P}_\lambda} \sum_{\mathcal{B} \text{ circum}} \mathbb{1}_{\mathcal{B}\cap\mathcal{P}_\lambda = \emptyset}]$$

Applying Mecke-Slivnyak theorem

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^n}{n!} \int_{x_1,\dots,x_n \in M} \sum_{\mathcal{B} \text{ circum}} e^{-\lambda \operatorname{vol}(\mathcal{B})} dx_1 \dots dx_n$$

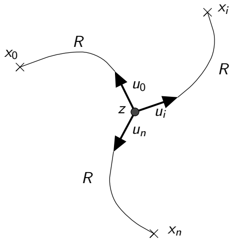
An expansion of the volume of a small geodesic ball on M is given by

$$\operatorname{vol}(\mathcal{B}(z,R)) = \kappa_n R^n \left(1 - \frac{\operatorname{Sc}(z)}{6(n+2)} R^2 + o(R^2)\right)$$

Blaschke Petkantschin change of variables

Let $\Phi : (R, z, u_0, \ldots, u_n) \mapsto (x_0, \ldots, x_n)$ be defined by

$$x_i = \exp_z(Ru_i)$$



Theorem

The Jacobian determinant J_Φ of Φ satisfies

$$J_\Phi(z, R, u_0, u_1, \ldots, u_n) = n!\Delta(u_0, \ldots, u_n) \prod_{i=0}^{n} \det M^{(i)}$$

Moreover, when R tends to 0,

$$J_\Phi(z, R, u_0, u_1, \ldots, u_n) = n!\Delta(u_0, u_1, \ldots, u_n) \left(R^{n^2-1} - \frac{\sum_{i=0}^{n} \mathrm{Ric}_z(u_i)}{6} R^{n^2+1} + o(R^{n^2+1}) \right)$$

Limit theorems

We derive limit theorems with a view to estimation of the curvature

- **Local geometry**: we focus on $\mathcal{B}(x_0, \lambda^{-\beta})$, with $0 < \beta < \frac{1}{n}$
- **Preserve the curvature**: we consider the variable

$$N = \sum_{x \in \mathcal{P}_\lambda \cap \mathcal{B}(x_0, \lambda^{-\beta})} N(C(x, \mathcal{P}_\lambda))$$

Limit theorems

Weak Law of Large Numbers

When $\lambda \to \infty$

$$\frac{1}{\lambda \operatorname{vol}(\mathcal{B}(x_0, \lambda^{-\beta}))}\mathbb{E}[N] = \mathbb{E}[N(\mathcal{C})] + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right) = E_n - \frac{\operatorname{Sc}(x_0)}{\lambda^{\frac{2}{n}}} C_n + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

Central Limit Theorem

When $\lambda \to \infty$,

$$\frac{N - \mathbb{E}[N]}{\sqrt{\operatorname{Var}(N)}} \to \mathcal{N}(0,1) \text{ in law}$$

Sketch of proof

Baldi-Rinott (89)

Let $\{X_{an}, a \in V_n\}$ r. v. having dependency graph $G_n = (V_n, E_n)$, $n \geq 1$. Let $S_n = \sum_{a \in V_n} X_{an}$, $\sigma_n^2 = \mathrm{Var}(S_n) < \infty$, D_n denote the maximal degree of G_n and suppose $|X_{an}| \leq B_n$ for some constant B_n a.s. for all $a \in V_n$. Then

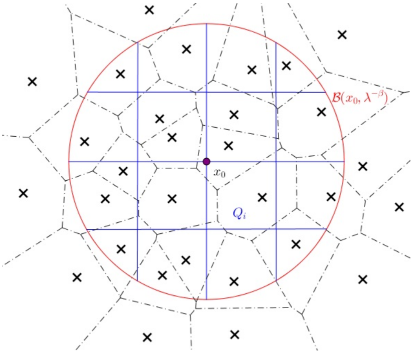
$$\left| \mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\sigma_n} \leq x \right) - \Phi(x) \right| \leq 32(1 + \sqrt{6}) \left(\frac{|V_n| D_n^2 B_n^3}{\sigma_n^3} \right)^{\frac{1}{2}}$$

1. We construct a dependency graph
2. We show that the bounds in Baldi-Rinott tends to 0

Dependency graph

- $\mathcal{B}(x_0, \lambda^{-\beta})$ is divided into $m_\lambda = \lambda^{n\alpha} \log(\lambda)^{-n} \lambda^{-n\beta}$ sets, Q_i, of volume $\lambda^{-n\alpha} \log(\lambda)^n$, $\frac{1}{n} > \alpha > \beta$.
- $$N_i = \sum_{x \in \mathcal{P}_\lambda \cap Q_i} N(C(x, \mathcal{P}_\lambda))$$
- all is considered "conditionally on A_λ" with

$$A_\lambda = \{\forall i, 1 \leq \mathcal{P}_\lambda(Q_i) \leq c\lambda\lambda^{-\alpha n} \log(\lambda)^n\}$$



Bound

- **Number of vertices:** $m_\lambda = \lambda^{n\alpha} \log(\lambda)^{-n} \lambda^{-n\beta}$
- **Maximal degree:** $D_\lambda \leq C_n$, constant
- **Bound of** N_i: $N_i \leq C'_n \lambda \lambda^{-\alpha n} \log(\lambda)^n$
- **Variance:** $\mathrm{Var}(N) \geq \lambda \lambda^{-n\beta}$
  (lower bound due to Last-Peccati-Schulte, 2014)

$$\left| \mathbb{P}\left(\frac{N - \mathbb{E}[N]}{\sqrt{\mathrm{Var}(N)}} \leq x \right) - \Phi(x) \right| \leq \log(\lambda)^n \lambda^{\frac{n\beta - 1}{4}} \to 0 \text{ when } \lambda \to \infty$$

Estimation of the scalar curvature

In order to estimate $\mathrm{Sc}(x_0)$, we define the estimator

$$\widehat{\mathrm{Sc}}_\lambda(x_0) = \frac{\lambda^{\frac{2}{n}}}{D_n}\left(E_n - \frac{1}{\lambda\,\mathrm{vol}(\mathcal{B}(x_0,\lambda^{-\beta}))}\sum_{x\in\mathcal{P}_\lambda\cap\mathcal{B}(x_0,\lambda^{-\beta})} N(C(x,\mathcal{P}_\lambda))\right)$$

Properties

When λ tends to ∞, $\widehat{\mathrm{Sc}}_\lambda(x_0)$ is

- asymptotically unbiased
- asymptotically normal
- convergent, for $n \geq 5$ and $\beta < \frac{1}{n} - \frac{4}{n^2}$

Euler characteristic and Gauss-Bonnet theorem

S compact surface without boundary

Gauss-Bonnet theorem

$$2\pi\chi(S) = \int_{x \in S} K(x)dx$$

For all graph on S,

$$\chi(S) = V - E + F$$

V: vertices, E: edges, F: faces

Euler characteristic and Gauss-Bonnet theorem

S compact surface without boundary

Gauss-Bonnet theorem

$$2\pi\chi(S) = \int_{x\in S} K(x)dx$$

For all random graph on S,

$$\chi(S) = \mathbb{E}[V] - \mathbb{E}[E] + \mathbb{E}[F]$$

V: vertices, E: edges, F: faces

Voronoi diagram

For any Voroni diagram,

- each vertex is in three cells
- each edge is in two cells

so $3V = 2E$

$$\chi(S) = \mathbb{E}[F] - \frac{1}{2}\mathbb{E}[V]$$

Computation of $\mathbb{E}[V]$ and $\mathbb{E}[F]$

$$3\mathbb{E}[V] = \mathbb{E}[\sum_{C \ cell} N(C)] = \lambda \int_{x\in S} \mathbb{E}[N(C(x, \mathcal{P}_\lambda \cup \{x\}))]dx$$

$$\mathbb{E}[N(C(x, \mathcal{P}_\lambda \cup \{x\}))] = 6 - \frac{3K(x)}{\pi\lambda} + o\left(\frac{1}{\lambda}\right)$$

$$\mathbb{E}[V] = 2\lambda \, \mathrm{vol}(S) - \frac{1}{\pi}\int_{x\in S} K(x)dx + o(1)$$

$$\mathbb{E}[F] = \lambda \, \mathrm{vol}(S)$$

$$\chi(S) = \frac{1}{2\pi}\int_{x\in S} K(x)dx$$

Take Home Message

- **We did it**
  - ↪ Link between mean number of vertices and scalar curvature
  - ↪ Limit theorems for the number of vertices
  - ↪ Simple probabilistic proof of Gauss-Bonnet theorem in dimension 2.
- **Perspectives:**
  - ↪ Study of $\widehat{\mathrm{Sc}}_\lambda(x_0)$
  - ↪ Limit theorems for other characteristics and estimation of other curvatures
  - ↪ Generalized Gauss-Bonnet theorem for manifolds of even dimension
  - ↪ ...

Thank you for your attention!