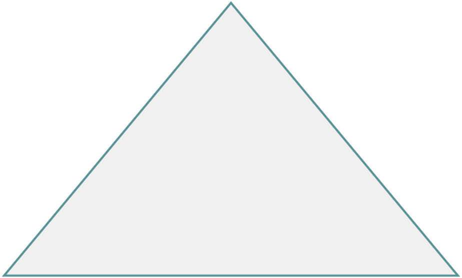
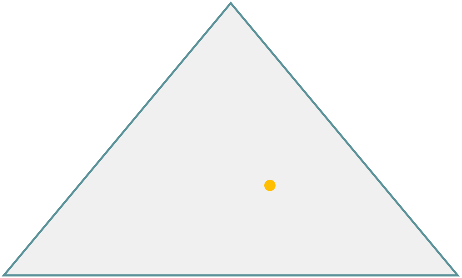
The moments of the number of vertices of a random polytope

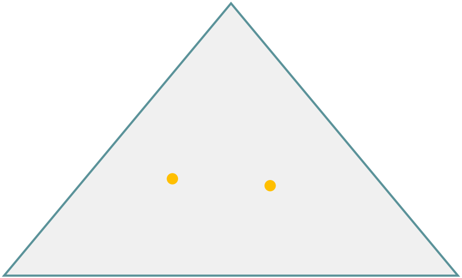
Christian Buchta

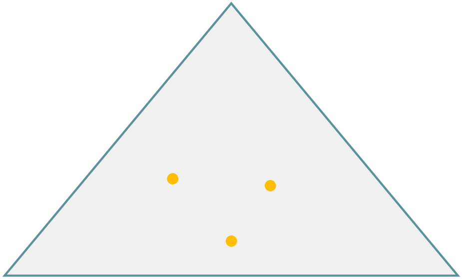
May 16, 2017

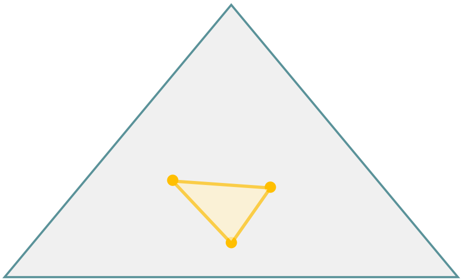
$$\mathbb{E}L_2 = \frac{1}{3}$$

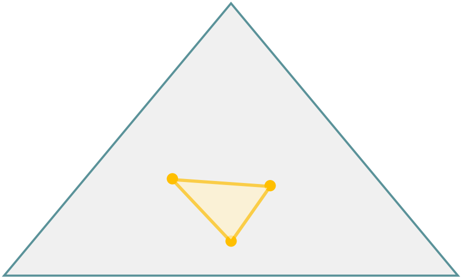




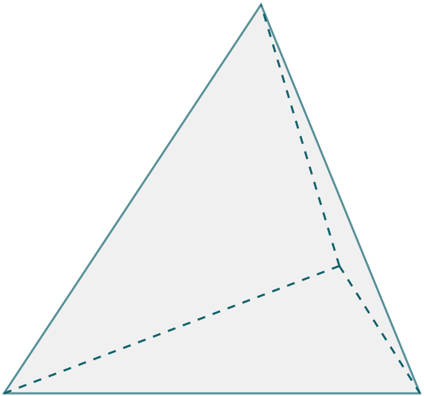


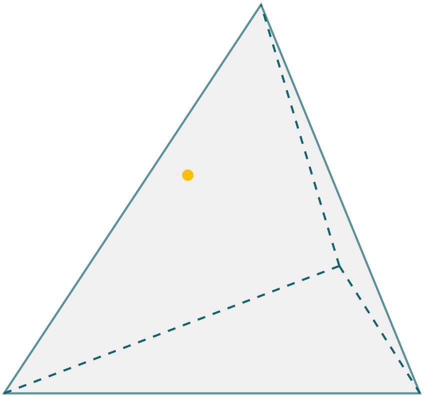


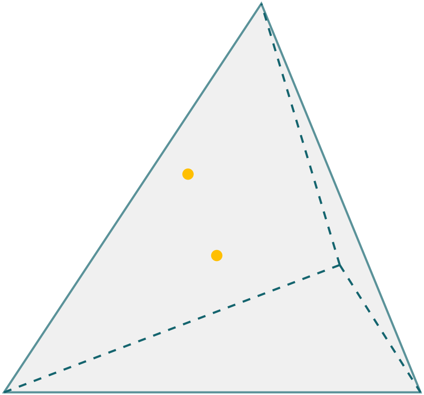


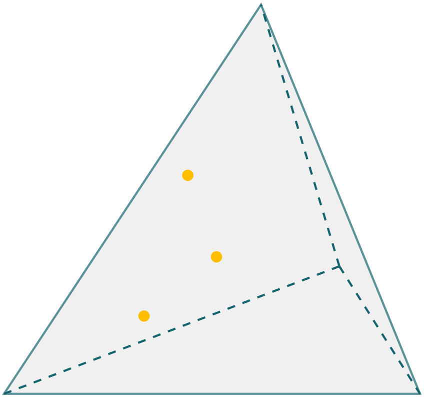


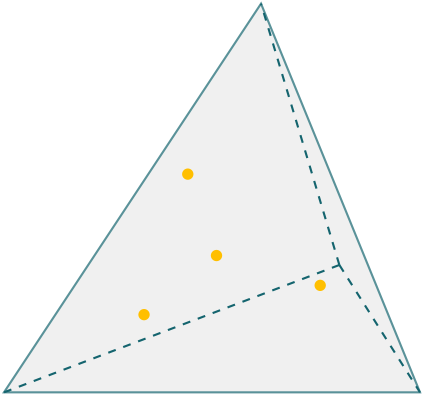
$$\mathbb{E}A_3 = \frac{1}{12}$$

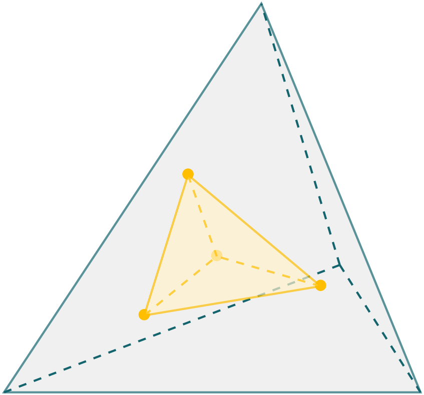


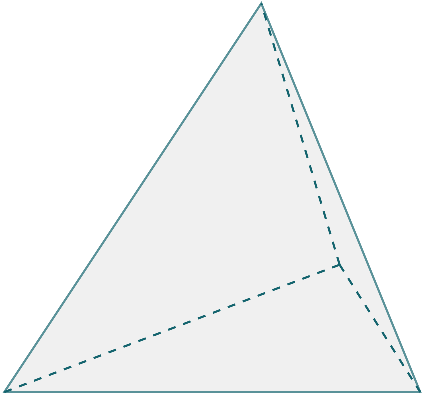


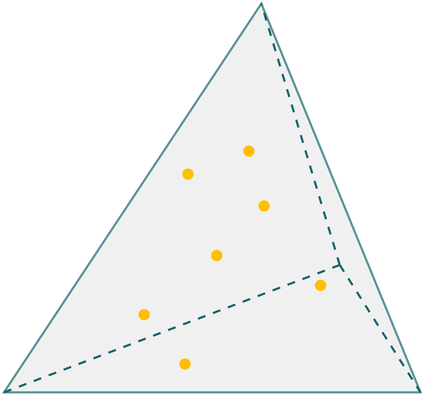


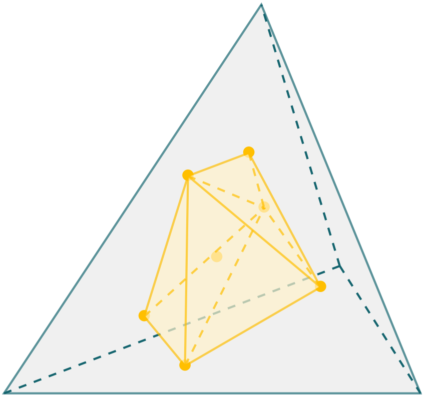












B. and Reitzner

$$
\begin{aligned}
\mathbb{E} V_n &= 1 - \frac{2}{n+1} - \frac{3(n-1)n}{4} \left[\frac{1}{(n+1)^3} + \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{1}{(k+3)^3} \right] \\
&\quad - \frac{9(n-1)n}{2} \sum_{\substack{j_1+\ldots+j_5=n-2 \\ k_1+k_2+k_3=4 \\ j_1,\ldots,j_5,k_1,k_2,k_3 \geq 0}} \binom{n-2}{j_1,\ldots,j_4} \binom{4}{k_1,k_2} 2^{k_2}\, 3^{\,j_2+j_3} \\
&\qquad \times B(j_2+2j_3+3j_4+3j_5+k_2+2k_3+1, 3j_1+2j_2+j_3+2k_1+k_2+1) \\
&\qquad \times B(n+1, j_5+k_3+1)\ B(2j_1+j_2+k_1+1, j_5+2) \\
&\qquad \times {}_3F_2(j_5+k_3, n+1, 2j_1+j_2+k_1+1; j_5+k_3+n+2, 2j_1+j_2+j_5+k_1+3; 1) \\
&\quad + 6(n-1)n \sum_{\substack{j_1+\ldots+j_5=n-2 \\ l_1+l_2=2 \\ l_3+l_4=2 \\ j_1,\ldots,j_5,l_1,l_2,l_3,l_4 \geq 0}} \binom{n-2}{j_1,\ldots,j_4} \binom{2}{l_1} \binom{2}{l_3} 3^{\,j_2+j_3} \\
&\qquad \times B(j_2+2j_3+3j_4+3j_5+l_2+l_4+3, 3j_1+2j_2+j_3+l_1+l_3+3) \\
&\qquad \times B(n+1, j_5+l_4+1)\ B(2j_1+j_2+l_1+1, j_5+3) \\
&\qquad \times {}_3F_2(j_5+l_4+1, n+1, 2j_1+j_2+l_1+1; j_5+l_4+n+2, 2j_1+j_2+j_5+l_1+4; 1).
\end{aligned}
$$

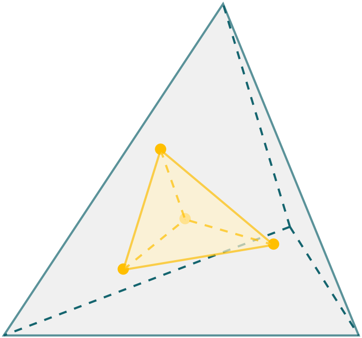
- "Classical" Hypergeometric Function:

$$
\begin{aligned}
{}_2F_1(\alpha_1,\alpha_2;\beta_1;x) &= \\
&= 1 + \frac{\alpha_1\alpha_2}{\beta_1}\frac{x}{1!} + \frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)}{\beta_1(\beta_1+1)}\frac{x^2}{2!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k}{(\beta_1)_k}\frac{x^k}{k!},
\end{aligned}
$$

  where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

- Generalized Hypergeometric Function:

$$
\begin{aligned}
{}_AF_B(\alpha_1,\alpha_2,\ldots,\alpha_A;\beta_1,\beta_2,\ldots,\beta_B;x) &= \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_A)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_B)_k}\frac{x^k}{k!}.
\end{aligned}
$$



$$\mathbb{E}V_4 = \frac{13}{720} - \frac{\pi^2}{15\,015}$$

UNIVERSITÄT SALZBURG

The first non-trivial values are:

$$
\begin{aligned}
\mathbb{E}V_4 &= \frac{13}{720} - \frac{\pi^2}{15015} && \approx 0.0173\\
\mathbb{E}V_5 &= \frac{13}{288} - \frac{\pi^2}{6006} && \approx 0.0434\\
\mathbb{E}V_6 &= \frac{127}{1680} - \frac{89\pi^2}{323323} && \approx 0.0728\\
\mathbb{E}V_7 &= \frac{307}{2880} - \frac{211\pi^2}{554268} && \approx 0.1028\\
\mathbb{E}V_8 &= \frac{41369}{302400} - \frac{22829\pi^2}{47805615} && \approx 0.1320\\
\mathbb{E}V_9 &= \frac{11129}{67200} - \frac{461\pi^2}{817190} && \approx 0.1600\\
\mathbb{E}V_{10} &= \frac{641303}{332640} - \frac{3058061\pi^2}{4775249765} && \approx 0.1864\\
\mathbb{E}V_{11} &= \frac{37723}{172800} - \frac{6445438\pi^2}{9116385915} && \approx 0.2113
\end{aligned}
$$

For the ellipsoid:

$$\mathbb{E}V_4 = \frac{9}{715} \approx 0.0126$$

$$\mathbb{E}V_5 = \frac{9}{286} \approx 0.0315$$

$$\mathbb{E}V_6 = \frac{3105}{58786} \approx 0.0528$$

$$\mathbb{E}V_7 = \frac{531}{7106} \approx 0.0747$$

$$\mathbb{E}V_8 = \frac{39333}{408595} \approx 0.0963$$

$$\mathbb{E}V_9 = \frac{36558}{312455} \approx 0.1170$$

$$\mathbb{E}V_{10} = \frac{404370}{2956811} \approx 0.1368$$

$$\mathbb{E}V_{11} = \frac{710478}{4569617} \approx 0.1555$$

The values are related by:

$$\mathbb{E}V_5 = \frac{5}{2}\,\mathbb{E}V_4,$$
$$\mathbb{E}V_7 = \frac{7}{2}\,\mathbb{E}V_6 - \frac{35}{4}\,\mathbb{E}V_4,$$
$$\mathbb{E}V_9 = \frac{9}{2}\,\mathbb{E}V_8 - 21\,\mathbb{E}V_6 + 63\,\mathbb{E}V_4,$$
$$\mathbb{E}V_{11} = \frac{11}{2}\,\mathbb{E}V_{10} - \frac{165}{4}\,\mathbb{E}V_8 + 231\,\mathbb{E}V_6 - \frac{2805}{4}\,\mathbb{E}V_4.$$

Identity of the values $\mathbb{E}V_n$ for dimension $d = 3$:

For $m = 2, 3, \ldots$

$$\mathbb{E}V_{2m+1} = \sum_{k=1}^{m-1} (2^{2k} - 1) \frac{B_{2k}}{k} \binom{2m+1}{2k-1} \mathbb{E}V_{2m-2k+2},$$

where the constants B_{2k} are the Bernoulli numbers

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, \ldots.$$

For arbitrary dimensions d and $m \in \mathbb{N}$:

$$\mathbb{E}V_{d+2m} = \sum_{k=1}^{m} \left(2^{2k} - 1\right) \frac{B_{2k}}{k} \binom{d+2m}{2k-1} \mathbb{E}V_{d+2m+1-2k}.$$

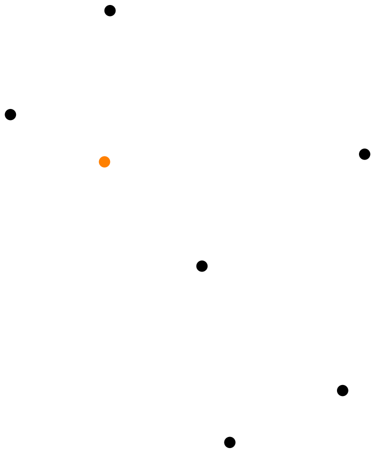
UNIVERSITÄT SALZBURG

For arbitrary dimensions d and $m \in \mathbb{N}$:

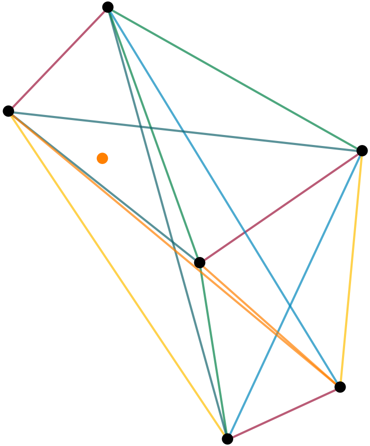
$$\mathbb{E}V_{d+2m} = \sum_{k=1}^{m} \left(2^{2k} - 1\right) \frac{B_{2k}}{k} \binom{d+2m}{2k-1} \mathbb{E}V_{d+2m+1-2k}.$$

Without Bernoulli numbers:

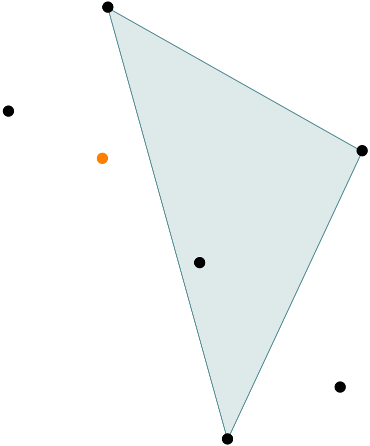
$$\mathbb{E}V_{d+2m} = \frac{1}{2} \sum_{k=1}^{2m-1} (-1)^{k-1} \binom{d+2m}{k} \mathbb{E}V_{d+2m-k}.$$



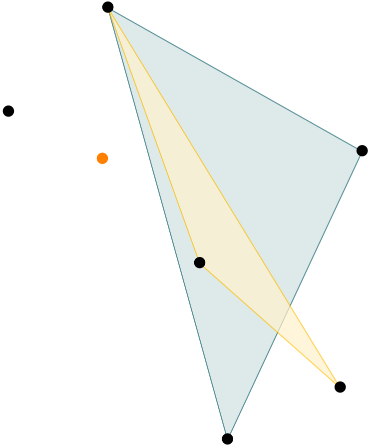
k	sign	$c_k(x)$
1	$+$	0
Σ		0



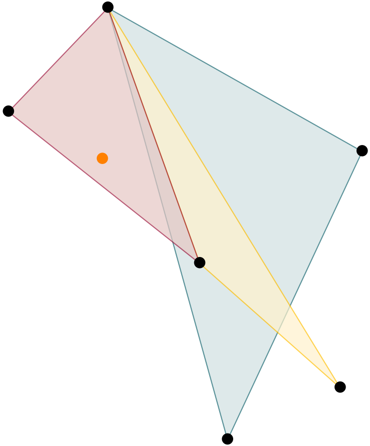
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
Σ		0



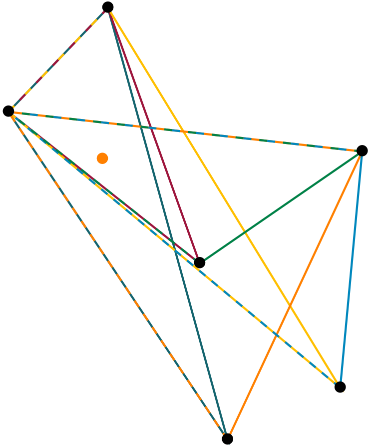
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
Σ		0



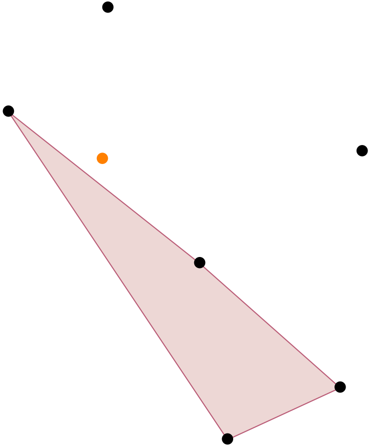
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
Σ		0



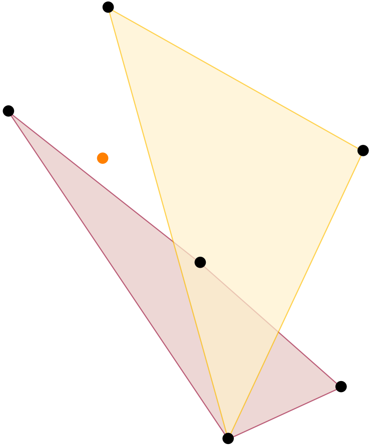
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
Σ		0



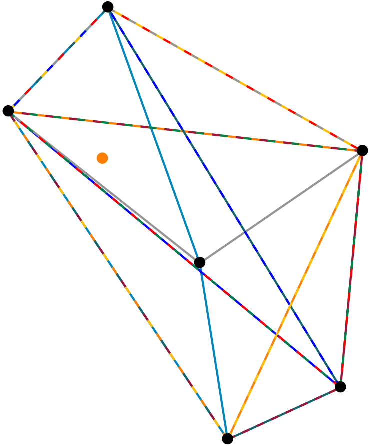
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
Σ		6



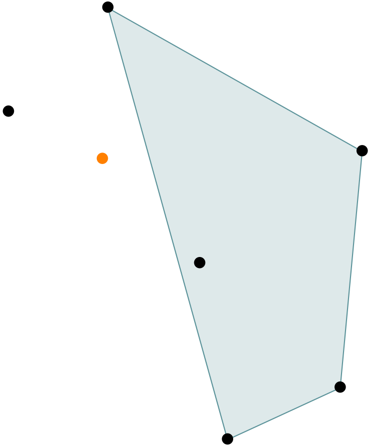
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
Σ		6



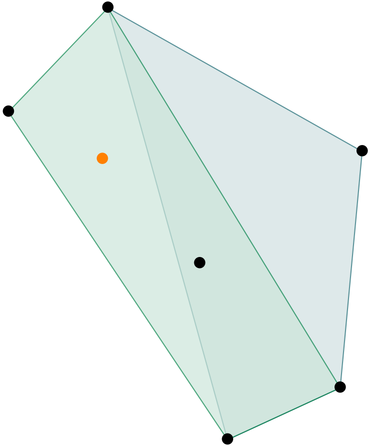
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
	Σ	6



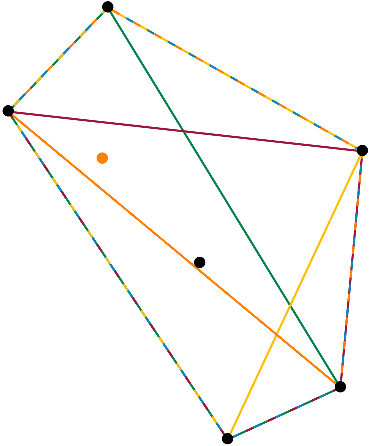
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
Σ		-3



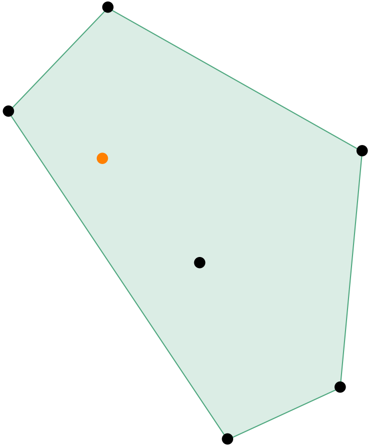
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
Σ		-3



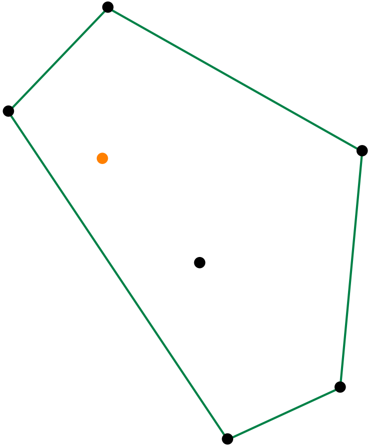
k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
Σ		-3



k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
5	$+$	5
	Σ	2



k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
5	$+$	5
Σ		2



k	sign	$c_k(x)$
1	$+$	0
2	$-$	0
3	$+$	6
4	$-$	9
5	$+$	5
6	$-$	1
	Σ	1

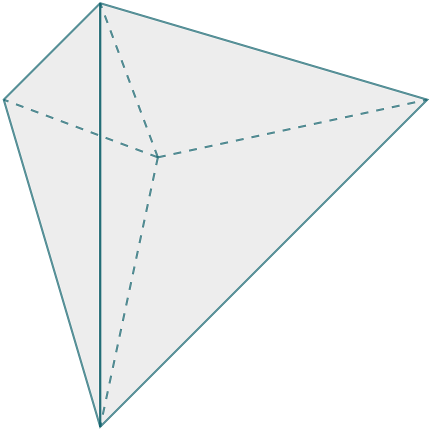
Cowan's conjecture

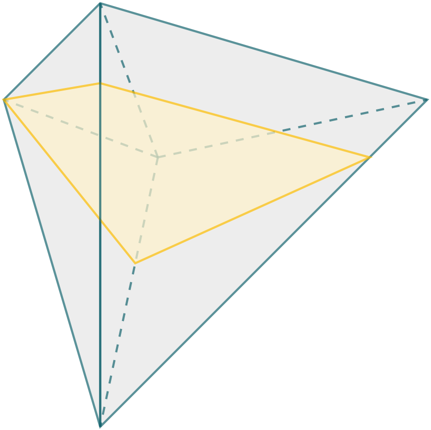
Let $x_1, \ldots, x_n$ be points in $\mathbb{R}^d$ (not necessarily distinct) with $K_n = \operatorname{conv}\{x_1, \ldots, x_n\}$, and let x be an additional point in $\mathbb{R}^d$.

$$c_k(x) := \#\{\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} : x \in \operatorname{conv}\{x_{i_1}, \ldots, x_{i_k}\}\},$$

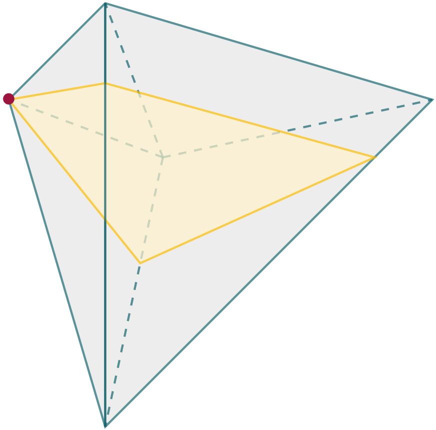
where $\#B$ is the number of elements in a set B. Then

$$\sum_{k=1}^{n} (-1)^{k-1} c_k(x) = \begin{cases} (-1)^{\dim K_n}, & \text{if } x \in \operatorname{relint} K_n, \\ 0, & \text{if } x \notin \operatorname{relint} K_n. \end{cases}$$

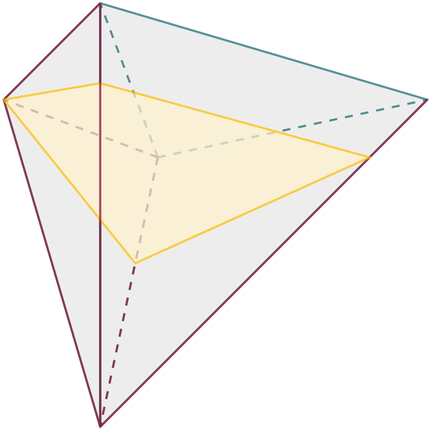




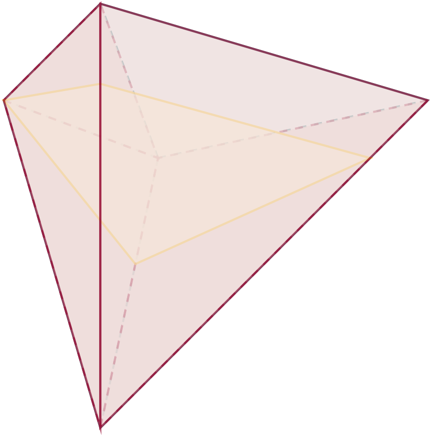
UNIVERSITÄT SALZBURG



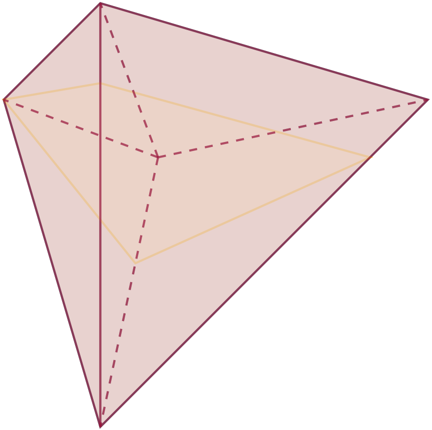
k	sign	a_k
0	$+$	1
Σ		1



k	sign	a_k
0	$+$	1
1	$-$	6
Σ		-5



k	sign	a_k
0	$+$	1
1	$-$	6
2	$+$	5
Σ		0

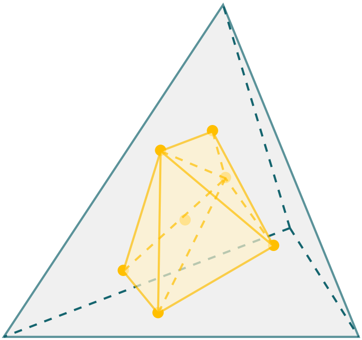


k	sign	a_k
0	$+$	1
1	$-$	6
2	$+$	5
3	$-$	1
Σ		-1

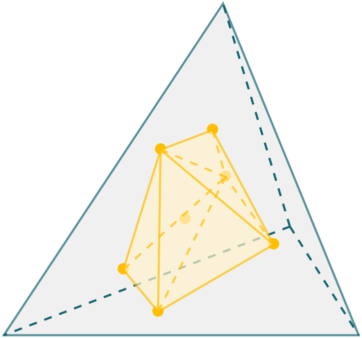
Kabluchko, Last, Zaporozhets 2017

Let T be a polytope in $\mathbb{R}^m$ with non-empty interior $\operatorname{int} T$. Let $L \subset \mathbb{R}^m$ be an affine subspace of dimension $m - d$. Denote by a_k the number of k-dimensional faces of T which are intersected by L, where $k = 0, \ldots, m$. Then

$$\sum_{k=0}^{m} (-1)^k a_k = \begin{cases} (-1)^d, & \text{if } L \cap \operatorname{int} T \neq \emptyset, \\ 0, & \text{if } L \cap \operatorname{int} T = \emptyset. \end{cases}$$



Efron's identity



$$\mathbb{E}N_{n+1} = (n+1)\left(1 - \frac{\mathbb{E}V_n}{\mathrm{vol} K}\right)$$

Efron's identity

$$\frac{\mathbb{E} V_n}{\mathrm{vol} K} = 1 - \frac{\mathbb{E} N_{n+1}}{n+1}$$

Efron's identity

$$\frac{\mathbb{E} V_n}{\mathrm{vol} K} = 1 - \frac{\mathbb{E} N_{n+1}}{n+1}$$

$$\frac{\mathbb{E} V_n^k}{(\mathrm{vol} K)^k} = \mathbb{E} \prod_{i=1}^{k} \left(1 - \frac{N_{n+k}}{n+i}\right)$$

UNIVERSITÄT SALZBURG

Efron's identity

$$\frac{\mathbb{E} V_n}{\mathrm{vol} K} = 1 - \frac{\mathbb{E} N_{n+1}}{n+1}$$

$$\frac{\mathbb{E} V_n^k}{(\mathrm{vol} K)^k} = \mathbb{E} \prod_{i=1}^{k} \left(1 - \frac{N_{n+k}}{n+i}\right)$$

$$\frac{\mathbb{E} V_n^k}{(\mathrm{vol} K)^k} = 1 - \left(\frac{1}{n+1} + \cdots + \frac{1}{n+k}\right) \mathbb{E} N_{n+k} + \cdots + (-1)^k \frac{1}{(n+1)\cdots(n+k)} \mathbb{E} N_{n+k}^k$$

For $n \in \mathbb{N}$ and $l = 1, \ldots, n$

$$p_l^{(n)} = (-1)^l \binom{n}{l} \sum_{j=1}^{l} (-1)^j \binom{l}{j} \frac{\mathbb{E} V_j^{n-j}}{(\mathrm{vol} K)^{n-j}}.$$

$p_l^{(n)}$ denotes the probability that $N_n = l$.

The matrix

$$\left(\binom{n-i}{n-j}\right)_{i,j=1,\dots n} = \begin{pmatrix} \binom{n-1}{n-1} & 0 & 0 & \cdots & 0 & 0 \\ \binom{n-1}{n-2} & \binom{n-2}{n-2} & 0 & \cdots & 0 & 0 \\ \binom{n-1}{n-3} & \binom{n-2}{n-3} & \binom{n-3}{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n-1}{1} & \binom{n-2}{1} & \binom{n-3}{1} & \cdots & \binom{1}{1} & 0 \\ \binom{n-1}{0} & \binom{n-2}{0} & \binom{n-3}{0} & \cdots & \binom{1}{0} & \binom{0}{0} \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} \binom{n-1}{n-1} & 0 & 0 & \cdots & 0 & 0 \\ -\binom{n-1}{n-2} & \binom{n-2}{n-2} & 0 & \cdots & 0 & 0 \\ \binom{n-1}{n-3} & -\binom{n-2}{n-3} & \binom{n-3}{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n\binom{n-1}{1} & (-1)^{n+1}\binom{n-2}{1} & (-1)^{n+2}\binom{n-3}{1} & \cdots & (-1)^{2n-2}\binom{1}{1} & 0 \\ (-1)^{n+1}\binom{n-1}{0} & (-1)^{n+2}\binom{n-2}{0} & (-1)^{n+3}\binom{n-3}{0} & \cdots & (-1)^{2n-1}\binom{1}{0} & (-1)^{2n}\binom{0}{0} \end{pmatrix}$$

$$= \left((-1)^{j+k} \binom{n-j}{n-k} \right)_{j,k=1,\ldots n}$$

UNIVERSITÄT SALZBURG

B. 2017+

For any real numbers $c_0, \ldots, c_n$

$$\sum_{l=0}^{n} c_l \, p_{n-l}^{(n)} = \sum_{j=0}^{n} \Delta^j c_0 \binom{n}{j} \frac{\mathbb{E} V_{n-j}^{j}}{(\mathrm{vol}\, K)^j},$$

B. 2017+

For any real numbers $c_0, \dots, c_n$

$$\sum_{l=0}^{n} c_l \, p_{n-l}^{(n)} = \sum_{j=0}^{n} \Delta^j c_0 \binom{n}{j} \frac{\mathbb{E} V_{n-j}^{j}}{(\mathrm{vol}\, K)^j},$$

where the difference operator is defined by $\Delta c_l = c_{l+1} - c_l$, i.e.

$$\Delta^j c_0 = \sum_{l=0}^{j} (-1)^{j+l} \binom{j}{l} c_l.$$

Special case: $c_l = n - l$

l	c_l	Δc_l	$\Delta^2 c_l$	$\cdots$	$\Delta^{n-2} c_l$	$\Delta^{n-1} c_l$	$\Delta^n c_l$
0	n	-1	0	$\cdots$	0	0	0
1	$n-1$	-1	0	$\cdots$	0	0	
2	$n-2$	-1	0	$\cdots$	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$n-2$	2	-1	0				
$n-1$	1	-1					
n	0						

Special case: $c_l = n - l$

l	c_l	Δc_l	$\Delta^2 c_l$	$\cdots$	$\Delta^{n-2} c_l$	$\Delta^{n-1} c_l$	$\Delta^n c_l$
0	n	-1	0	$\cdots$	0	0	0
1	$n-1$	-1	0	$\cdots$	0	0	
2	$n-2$	-1	0	$\cdots$	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$n-2$	2	-1	0				
$n-1$	1	-1					
n	0						

$$\mathbb{E}N_n = \sum_{l=0}^{n} c_l p_{n-l}^{(n)} = \sum_{j=0}^{n} \Delta^j c_0 \binom{n}{j} \frac{\mathbb{E}V_{n-j}^j}{(\mathrm{vol}K)^j}$$

$$= n + (-1)\, n \frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K} = n\left(1 - \frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K}\right)$$

Special case: $c_l = (n-l)^2$

l	c_l	Δc_l	$\Delta^2 c_l$	$\Delta^3 c_l$	$\cdots$	$\Delta^{n-2} c_l$	$\Delta^{n-1} c_l$	$\Delta^n c_l$
0	n^2	$-2n+1$	2	0	$\cdots$	0	0	0
1	$(n-1)^2$	$-2n+3$	2	0	$\cdots$	0	0	
2	$(n-2)^2$	$-2n+5$	2	0	$\cdots$	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$n-3$	9	-5	2	0				
$n-2$	4	-3	2					
$n-1$	1	-1						
n	0							

UNIVERSITÄT SALZBURG

Special case: $c_l = (n-l)^2$

l	c_l	Δc_l	$\Delta^2 c_l$	$\Delta^3 c_l$	$\cdots$	$\Delta^{n-2} c_l$	$\Delta^{n-1} c_l$	$\Delta^n c_l$
0	n^2	$-2n+1$	2	0	$\cdots$	0	0	0
1	$(n-1)^2$	$-2n+3$	2	0	$\cdots$	0	0	
2	$(n-2)^2$	$-2n+5$	2	0	$\cdots$	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$n-3$	9	-5	2	0				
$n-2$	4	-3	2					
$n-1$	1	-1						
n	0							

$$\mathbb{E} N_n^2 = \sum_{l=0}^{n} c_l p_{n-l}^{(n)} = \sum_{j=0}^{n} \Delta^j c_0 \binom{n}{j} \frac{\mathbb{E} V_{n-j}^j}{(\mathrm{vol} K)^j}$$

$$= n^2 + (-2n+1)\, n \frac{\mathbb{E} V_{n-1}}{\mathrm{vol} K} + 2 \frac{n(n-1)}{2} \frac{\mathbb{E} V_{n-2}^2}{(\mathrm{vol} K)^2}$$

B. 2017+

There are polynomials $p_j^{(k)}(n)$ of degree k in n such that

$$\mathbb{E} N_n^k = \sum_{j=0}^{k} p_j^{(k)}(n) \frac{\mathbb{E} V_{n-j}^j}{(\operatorname{vol} K)^j}.$$

B. 2017+

There are polynomials $p_j^{(k)}(n)$ of degree k in n such that

$$\mathbb{E} N_n^k = \sum_{j=0}^{k} p_j^{(k)}(n) \frac{\mathbb{E} V_{n-j}^j}{(\operatorname{vol} K)^j}.$$

Denoting the Stirling numbers of the second kind by $\left\{ {k \atop i} \right\}$, these polynomials are given by

$$p_j^{(k)}(n) = (-1)^j \sum_{i=j}^{k} \binom{i}{j} \left\{ {k \atop i} \right\} n_{(i)},$$

with $n_{(i)} = \frac{n!}{(n-i)!}$.

Stirling numbers of the second kind

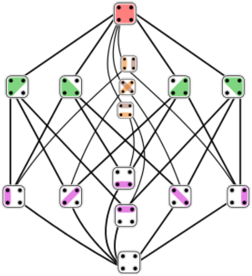


Figure : Wikipedia

$$\left\{ {4 \atop 1} \right\} = 1 \quad \left\{ {4 \atop 2} \right\} = 7 \quad \left\{ {4 \atop 3} \right\} = 6 \quad \left\{ {4 \atop 4} \right\} = 1$$

Stirling numbers of the second kind

- Recurrence relation:

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$$

Stirling numbers of the second kind

- Recurrence relation:

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$$

- With $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k = x^n$$

B. 2017+

Alternatively, there are polynomials $q_j^{(k)}(n)$ of degree $k-j$ in n, such that

$$\mathbb{E} N_n^k = \sum_{j=0}^{k} q_j^{(k)}(n)\, n_{(j)} \frac{\mathbb{E} V_{n-j}^j}{(\operatorname{vol} K)^j},$$

B. 2017+

Alternatively, there are polynomials $q_j^{(k)}(n)$ of degree $k-j$ in n, such that

$$\mathbb{E} N_n^k = \sum_{j=0}^{k} q_j^{(k)}(n)\, n_{(j)} \frac{\mathbb{E} V_{n-j}^j}{(\mathrm{vol}\, K)^j},$$

with

$$q_j^{(k)}(n) = \sum_{i=j}^{k} (-1)^i \binom{k}{i} \left\{ {i \atop j} \right\} n^{k-i}.$$

UNIVERSITÄT SALZBURG

$$\mathbb{E} N_n^k = \sum_{j=0}^{k} p_j^{(k)}(n) \frac{\mathbb{E} V_{n-j}^j}{(\mathrm{vol}\, K)^j} = \sum_{j=0}^{k} q_j^{(k)}(n)\, n_{(j)} \frac{\mathbb{E} V_{n-j}^j}{(\mathrm{vol}\, K)^j}$$

$$\mathbb{E}N_n^k = \sum_{j=0}^{k} p_j^{(k)}(n) \frac{\mathbb{E}V_{n-j}^j}{(\mathrm{vol}\,K)^j} = \sum_{j=0}^{k} q_j^{(k)}(n)\, n_{(j)} \frac{\mathbb{E}V_{n-j}^j}{(\mathrm{vol}\,K)^j}$$

$$\begin{aligned}
\mathbb{E}N_n &= n + (-n)\frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K} \\
&= n + (-1)n_{(1)}\frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K} \\
\mathbb{E}N_n^2 &= n^2 + (-2n^2 + n)\frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K} + (n^2 - n)\frac{\mathbb{E}V_{n-2}^2}{(\mathrm{vol}K)^2} \\
&= n^2 + (-2n + 1)n_{(1)}\frac{\mathbb{E}V_{n-1}}{\mathrm{vol}K} + (1)n_{(2)}\frac{\mathbb{E}V_{n-2}^2}{(\mathrm{vol}K)^2}
\end{aligned}$$

Thank you for your attention!

Stirling numbers of the first kind

- Recurrence relation:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

- With $(x)^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$

$$(x)^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$