Maximal degree in a Poisson-Delaunay graph

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joint work with

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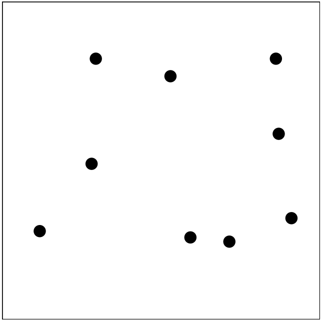

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   - stationary
   - intensity 1

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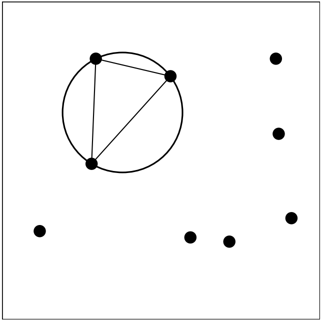

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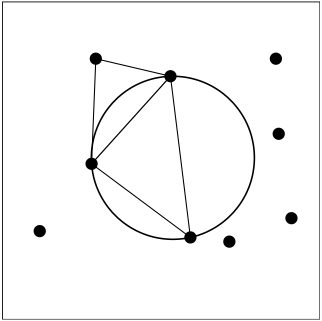

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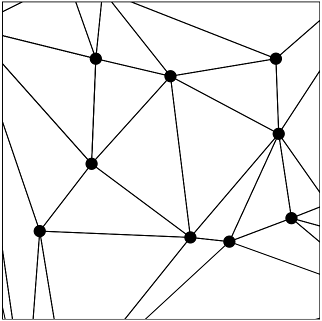

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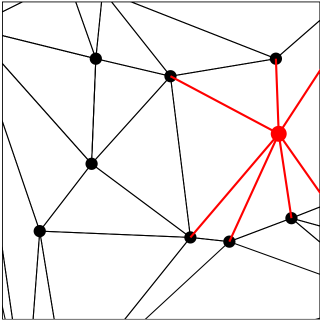

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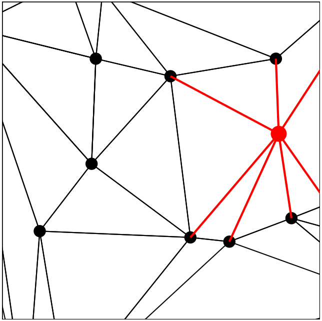

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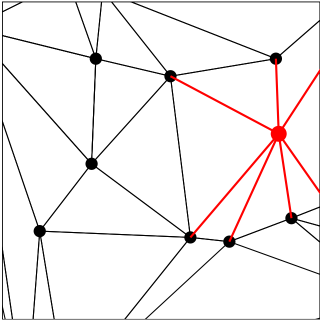

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Similar results from Penrose about random geometric graphs.

Additional difficulties:

- typical degree $\neq$ Poisson
- local condition

Results/Conjecture

$\Delta_n = \max_{x \in \eta \cap W_n} \deg(x)$ ... maximal degree in a window of volume n.

Theorem [Bern, Eppstein, Yao, 1991]

$$\mathbb{E}\Delta_n = \Theta\left(\frac{logn}{\log\log n}\right)$$

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Conjecture (... but soon to be a Theorem) [B., Chenavier]

There exists a deterministic sequence $(I_n)_{n \geq 1}$ such that

1. $\mathbb{P}(\Delta_n \in \{I_n, I_n + 1\}) \xrightarrow[n \to \infty]{} 1$;
2. $I_n \sim \frac{1}{2} \cdot \frac{\log n}{\log\log n}$.

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- Points with high degree are isolated
- Dependency graph

typical degree

$\deg_{\mathrm{typ}}$ ... typical degree

$$\mathbb{P}(\deg_{\mathrm{typ}} = k) := \frac{1}{\mathrm{Vol}(W)} \mathbb{E}\left[\sum_{x \in \eta \cap W} \mathbb{1}(\deg(x) = k)\right], \qquad \text{where } W \subset \mathbb{R}^2.$$

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Theorem : [Hilhorst '05]

$$\mathbb{P}(\deg_{\mathrm{typ}} = n) \sim n^{-2n} n^{-1/2} c,$$

as $n \to \infty$, where $c = \frac{e^2}{4\sqrt{\pi}} \prod_{q=1}^{\infty} (1 - \frac{1}{q} + \frac{4}{q^4})^{-1}$

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More results of this flavor for hyperplane tesselation:

- [Hilhorst, Calka, '08] number of facets of zero cell (in $\mathbb{R}^2$)
- [B., Calka, Reitzner, '17] number of facets of typical cell (in $\mathbb{R}^d$)
- Ph.D. thesis of B. ('16)

Deterministic sequence l_n

$G : \mathbb{R}_+ \to [0,1]$ ... continuous decreasing with $G(k) = \mathbb{P}(\deg_{\mathrm{typ}} > k)$
$l_n =$ closest integer from $G^{-1}\left(\frac{1}{n}\right)$

Lemma

As $n \to \infty$,

1. $n\,\mathbb{P}(\deg_{\mathrm{typ}} \geq l_n) \to \infty$;
2. $n^{\alpha}\,\mathbb{P}(\deg_{\mathrm{typ}} \geq l_n) \to 0$, for any $\alpha < 1$;

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From 3 we get

$$\mathbb{P}(\Delta_n \geq I_n + 2) \leq \mathbb{E}\left(\sum_{x \in \eta \cap W_n} \mathbb{1}(\deg(x) \geq I_n + 2)\right) = n\,\mathbb{P}(\deg_{\text{typ}} \geq I_n + 2) \longrightarrow 0$$

General idea for the inequality $\mathbb{P}(\Delta_n < I_n) \to 0$, as $n \to \infty$

What would happen if the degrees at each points of the process would be independent?

General idea for the inequality $\mathbb{P}(\Delta_n < l_n) \to 0$, as $n \to \infty$

What would happen if the degrees at each points of the process would be independent? Life would be easy! We would get

$$\begin{aligned}\mathbb{P}(\Delta_n < l_n) &= \mathbb{P}\left(\sum_{x\in\eta\cap W_n} \mathbb{1}(\deg(x) \geq l_n) = 0\right) \\ &= \mathbb{P}(Bin(Po(n)\,,\,\mathbb{P}(\deg_{\mathrm{typ}} \geq l_n)) = 0) \\ &\to 0 \qquad \text{since } n\,\mathbb{P}(\deg_{\mathrm{typ}} \geq l_n) \to \infty.\end{aligned}$$

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(difficult!) Local condition

$$n \int_{B(0, \log n)} \mathbb{P}(\deg_{\eta \cup \{0,y\}}(0) \geq l_n\,,\, \deg_{\eta \cup \{0,y\}}(y) \geq l_n)\,\mathrm{d}y \to 0.$$

Dependency graph

We discretize the window of volume n into $\sqrt{\frac{n}{\log n}} \times \sqrt{\frac{n}{\log n}}$ smaller squares of volume $\log n$.

Set (V, E) the graph with

- vertex set $V = \left\{1, \ldots, \sqrt{\frac{n}{\log n}}\right\} \times \left\{1, \ldots, \sqrt{\frac{n}{\log n}}\right\}$
- edges $\{(i,j),(i',j')\}$ when $\max(|i'-i|, |j'-j|) \leq 4$

Set $M_{i,j} := \max(\deg(x) : x \in \mathrm{Square}_{i,j})$, with $i, j = 1, \ldots, \sqrt{\frac{n}{\log n}}$.

Lemma

(V, E) is a dependency graph for the random variables $M_{i,j}$, i.e. if $V_1, V_2 \subset V$ such that $E \cap (V_1 \times V_2) = \emptyset$ then $\sigma(M_{i,j} : (i,j) \in V_1) \perp\!\!\!\perp \sigma(M_{i,j} : (i,j) \in V_2)$.

Dependency graph $\Rightarrow$ Poisson approximation

[Arratia, Goldstein, Gordon, 1990]

$$\sup_{S \subset \mathbb{N}} \left| \mathbb{P}\left(\sum_{(i,j) \in V} \mathbb{1}(M_{i,j} \geq l_n) \in S \right) - \mathbb{P}(Po(\mu) \in S) \right| \leq 2D \cdot |V| \cdot (A+B)$$

where D is the degree of the dependency graph, $|V| = \frac{n}{\log n}$,

$$\mu := \mathbb{E}\left[\sum_{(i,j) \in V} \mathbb{1}(M_{i,j} \geq l_n) \right]$$

$$A := \sup_{(i,j) \in V} \mathbb{P}(M_{i,j} \geq l_n)^2$$

and

$$B := \sup_{\{(i,j),(i',j')\} \in E} \mathbb{P}(M_{i,j} \geq l_n, M_{i',j'} \geq l_n)$$

Dependency graph $\Rightarrow$ Poisson approximation

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$$\sup_{S\subset\mathbb{N}}\left|\mathbb{P}\left(\sum_{(i,j)\in V}\mathbb{1}(M_{i,j}\geq l_n)\in S\right)-\mathbb{P}(Po(\mu)\in S)\right|\leq 2D\cdot|V|\cdot(A+B)$$

where D is the degree of the dependency graph, $|V|=\frac{n}{\log n}$,

$$\mu:=\mathbb{E}\left[\sum_{(i,j)\in V}\mathbb{1}(M_{i,j}\geq l_n)\right]\to\infty$$

$$A:=\sup_{(i,j)\in V}\mathbb{P}(M_{i,j}\geq l_n)^2=o\left(\frac{\log n}{n}\right)$$

and

$$B:=\sup_{\{(i,j),(i',j')\}\in E}\mathbb{P}(M_{i,j}\geq l_n,M_{i',j'}\geq l_n)=o\left(\frac{\log n}{n}\right)$$

Dependency graph ⇒ Poisson approximation

$$\sup_{S \subset \mathbb{N}} \left| \mathbb{P}\left(\sum_{(i,j) \in V} \mathbb{1}(M_{i,j} \geq l_n) \in S \right) - \mathbb{P}(Po(\mu) \in S) \right| \leq o(1),$$

with $\mu \to \infty$. Applied to $S = \{0\}$, this gives

$$\mathbb{P}\left(\sum_{(i,j) \in V} \mathbb{1}(M_{i,j} \geq l_n) = 0 \right) \to 0,$$

and thus

$$\mathbb{P}(\Delta_n \geq l_n) = 1 - \mathbb{P}\left(\sum_{(i,j) \in V} \mathbb{1}(M_{i,j} \geq l_n) = 0 \right) \to 1.$$

Thank you !