Mean geometry of excursion sets for 2D random fields: application to shot noise fields

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Shot noise random fields

A **(Poisson) shot noise random field** is a random field $(X(x))_{x\in\mathbb{R}^d}$ given by

$$\forall x \in \mathbb{R}^d, \quad X(x) = \sum_{i\in I} g_{m_i}(x - x_i),$$

where

- $\{x_i\}_{i\in I}$ is a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$,
- $\{m_i\}_{i\in I}$ are independent « marks » with distribution $F(dm)$ on $\mathbb{R}^k$, and independent of $\{x_i\}_{i\in I}$.
- The functions g_m are real-valued deterministic functions, called **spot functions**, such that

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^d} |g_m(y)|\, dy\, F(dm) < +\infty.$$

Shot noise random fields

There are many mathematical studies of this random field : S.O. Rice (1944), Papoulis (1971), Bar David and Nemirovsky (1972), Heinrich and Schmidt (1985), Baccelli and Blaszczyszyn (2001), etc...

It is also used in texture synthesis or image modeling : J. van Wijk (*Computer Graphics*, 1991), Lagae et al. (*SIGGRAPH*, 2009), Galerne et al. (*SIGGRAPH*, 2012) A. Srivastava, X. Liu et U. Grenander (IEEE PAMI 2002)...

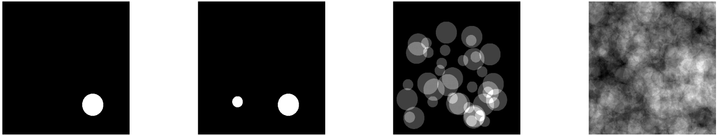
For sake of simplicity, in this talk we consider $d = 2$, $k = 1$ and a single $L^1(\mathbb{R}^2)$ function g randomly dilated : F is a probability measure on $(0, +\infty)$ and for $m > 0$

$$g_m(x) = g(x/m).$$

Example : disk with random radius

Let $d = 2$, $g = \mathbf{1}_D$, $U = (0, T)^2$ and consider random disk $g_m = \mathbf{1}_{D_m}$ of radius $m = m_1$ or $m = m_2$ with $0 < m_1 < m_2$ (each with probability $1/2$) with intensity $\lambda > 0$

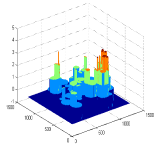
- The number of centers in $(-m_2, T + m_2)^2$ is a Poisson random variable of parameter $\lambda(T + 2m_2)^2 \longrightarrow n$
- The centers $x_1, \ldots, x_n$ are thrown uniformly, independently on $(-m_2, T + m_2)^2$
- The radius $R_1, \ldots, R_n$ are attached to each center by flipping a coin to choose between m_1 or m_2.



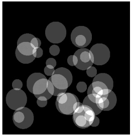
Excursion set

We consider the excursion set or the level set of level $t \in \mathbb{R}$ of X in U defined by

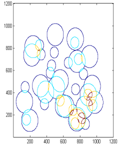
$$E_X(t, U) := \{x \in U; X(x) \geq t\}.$$



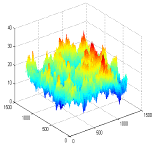
view 3D

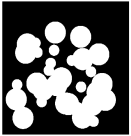


view 2D

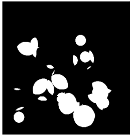


level lines



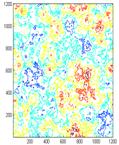


$t = 0.5$



$t = 1.5$

$t = 2.5$



Main questions

What can be said about "mean" geometry of excursion sets ? Area ? Perimeter ? Euler Characteristic= # connected components – # holes ?

Known results for

- Boolean model : Mecke (2001), Mecke, Wagner (1991)
- Smooth Gaussian random fields : Adler (2000), Adler, Taylor (2007), Azaïs, Wschebor (2009), ...
- High levels for infinitely divisible smooth random fields : Adler, Samorodnitsky, Taylor (2010,2013),...

Two different frameworks

1. Elementary : g is piecewize constant with compact support
2. Smooth : g is at least C^2

Statistical properties of shot noise random fields

The shot noise random field is given by

$$X(x) = \sum_{i \in I} g_{m_i}(x - x_i) = \sum_{i \in I} g\left(\frac{x - x_i}{m_i}\right), \forall x \in \mathbb{R}^2$$

where $\{x_i, m_i\}_{i \in I}$ is a marked Poisson point process of intensity $\lambda dx\, F(dm)$ on $\mathbb{R}^2 \times \mathbb{R}^+$ with $g \in L^1(\mathbb{R}^2)$ and $\mathbb{E}(R^2) < +\infty$ if $R \sim F$.

- The random field is stationary : its distribution is invariant by translation : $(X(x + x_0))_x \stackrel{d}{=} (X(x))_x$
- when g is radial the random field is isotropic : its distribution is invariant by orthogonal transformation : $(X(Ax))_x \stackrel{d}{=} (X(x))_x$ for all $A \in O_2(\mathbb{R})$
- The expectation (mean value) of X is given by

$$\mathbb{E}X(x) = \mathbb{E}X(0) = \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} g_m(y)\, dy\, F(dm) = \lambda \mathbb{E}(R^2) \int g.$$

Statistical properties of shot noise random fields

- If $\int_{\mathbb{R}^2} g(y)^2 dy < +\infty$, then X has second-order moments

$$\begin{aligned} \mathrm{Cov}(X(z), X(z+x)) &= \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} g_m(y) g_m(y-x)\, dy\, F(dm) \\ &= \lambda \rho(x). \end{aligned}$$

  In particular

$$\mathrm{Var}(X(x)) = \mathrm{Var}(X(0)) = \lambda \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} g_m(y)^2\, dy\, F(dm) = \lambda \mathbb{E}(R^2) \int g^2.$$

- When moreover the intensity λ goes to $+\infty$, the normalized random field

$$Z(x) = \frac{X(x) - \mathbb{E}(X(x))}{\sqrt{\lambda}}$$

  converges (f.d.d.) to a stationary centered Gaussian field with covariance ρ.

Statistical properties of shot noise random fields

When $g = \mathbf{1}_D$, the shot noise field has integer value and for any $t \in (0,1]$, $\mathbf{1}_{X \geq t}$ is a Boolean model. Moreover

- $\mathbb{E}X(x) = \mathbb{E}X(0) = \lambda \mathbb{E}(R^2)\mathcal{L}(D) = \lambda \mathbb{E}(\mathcal{L}(D_R)) = \lambda \bar{a}$
- $\mathrm{Cov}(X(z), X(z+x)) = \lambda \rho(x)$ with $\rho(x) = \mathbb{E}\left(\mathcal{L}(D_R \cap D_R + x)\right)$
- Since the characteristic function of $X(0)$ (or any $X(x)$) is given by :

  $$\forall u \in \mathbb{R}, \quad \mathbb{E}(e^{iuX(0)}) = \exp\left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} [e^{iug_m(y)} - 1] \, dy \, F(dm)\right),$$

  We get that $X(0) \sim \mathcal{P}(\lambda \bar{a})$.

Geometry of excursion sets

We consider the excursion set of level $t \in \mathbb{R}$ of X in $U = (0, T)^2$ defined by

$$E_X(t, U) := \{x \in U; X(x) \geq t\}.$$

The **mean area** is therefore (by stationarity of X)

$$\begin{aligned} \mathbb{E}\left(\mathcal{L}(E_X(t, U))\right) &= \int_U \mathbb{E}\left(\mathbf{1}_{X(x) > t}\right) dx \\ &= \mathcal{L}(U)\mathbb{P}(X(0) > t). \end{aligned}$$

The **mean perimeter** corresponds to

$$\mathbb{E}\left(\mathcal{H}^1(\partial E_X(t, U) \cap U)\right),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure of the length of plane curves.

The **mean Euler characteristic** corresponds to

$$\mathbb{E}\left(\chi(E_X(t, U))\right),$$

where $\chi(A)$ denotes the Euler characteristic of a set A.

Weak framework for function of special bounded variation

Let f be a real function piecewize constant with special bounded variation in U and approximate discontinuity set $\mathcal{S}_f$.
By Federer-Vol'pert Theorem,

- $\mathcal{S}_f$ is a $\mathcal{L}$-negligible Borel set, countably $\mathcal{H}^1$-rectifiable ;
- the distributional derivative of f is given by

  $$Df := (f^+ - f^-)\nu_f \mathcal{H}^1 \angle J_f,$$

  where $J_f \subset S_f$ is the $\mathcal{H}^1$-rectifiable set of *approximate jump points* of f : $\exists f^-(x) < f^+(x)$ and $\nu_f(x) \in S^{n-1}$ with

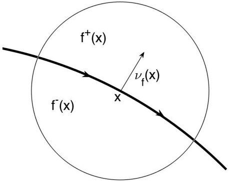
  $$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho^\pm(x,\nu_f(x))} |f(y) - f^\pm(x)| dx = 0.$$

  where $B_\rho^+(x,\nu) = \{y \in B_\rho(x); (y-x)\cdot\nu > 0\}$, resp. $B_\rho^-(x,\nu) = \{y \in B_\rho(x); (y-x)\cdot\nu > 0\}$.
- $\mathcal{H}^1(S_f \setminus J_f) = 0$.

Elementary function

The function f is said elementary function if f is a piecewise constant function with special bounded variation in U taking a finite number of values and if $\mathcal{S}_f$ corresponds to the discontinuity set of f in U and can be decomposed as

$$\mathcal{S}_f = \mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{I}_f, \text{ where :}$$

- 

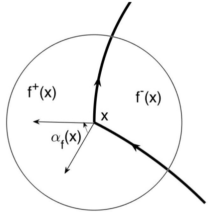
  $\mathcal{R}_f = J_f$ is a finite disjoint union of C^2 simple curves (possibly closed) with finite length and finite total curvature

ie $\kappa_f \in L^1(\mathcal{R}_f, \mathcal{H}^1)$, where the **signed curvature** $\kappa_f(x)$ of $\mathcal{R}_f$ at $x = \gamma(s)$ is given by

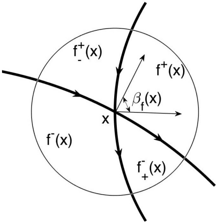
$$\kappa_f(x) = \langle \gamma''(s), \gamma'(s)^\perp \rangle = \langle \gamma''(s), \nu_f(x) \rangle,$$

for a local arc-length parametrization γ.

Case of elementary function

- 

  $\mathcal{C}_f$ is a finite set of corner points with turning angle
  $$\alpha_f(x) \in (-\pi, \pi)$$

- 

  $\mathcal{I}_f$ is a finite set of intersection points ie points x s.t. $\{x\} = \gamma_1 \cap \gamma_2$ with γ_1, γ_2 C^2 simple curves in $\mathcal{S}_f$ with at least 3 $\neq$ values
  $$f^-(x) \leq f_-^+(x), f_+^-(x) \leq f^+(x),$$

and
$$\beta_f(x) := d_{S^1}(\nu_{\gamma_1}(x), \nu_{\gamma_2}(x)) \in (0, \pi).$$

Closed set via elementary function

Proposition : Let $E \subset U$ be a regular region s.t. $f := \mathbf{1}_E$ is an elementary function in U. Then, $S_f = \partial E$ and

- the perimeter of E defined by

  $$\mathrm{Per}(E, U) := \|Df\|(U) = \sup\{\int_U f \mathrm{div}\varphi dx \,|\, \varphi \in C_c^1(U, \mathbb{R}^2), \|\varphi\|_\infty\} < +\infty,$$

  corresponds to

  $$\mathcal{H}^1(\mathcal{R}_f) = \mathcal{H}^1(\partial E).$$

- By Gauss-Bonnet Theorem, the Euler characteristic of E is given by

  $$\chi(E) = \frac{1}{2\pi}\mathrm{TC}(\partial E, U),$$

  where the total curvature of ∂E is equal to

  $$\mathrm{TC}(\partial E, U) = \int_{\mathcal{R}_f} \kappa_f(x)\mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f} \alpha_f(x).$$

Weak framework

Let f be an elementary function in U and set

$$E_f(t;U) = \{x \in U; f(x) \geq t\}.$$

Remark that when $t < \min_U f$ or $t > \max_U f$, then $\partial E_f(t;U) \cap U = \emptyset$.
We consider for any h bounded continuous function on $\mathbb{R}$

- the **level perimeter integral** of f

$$\mathrm{LP}_f(h,U) = \int_{\mathbb{R}} h(t)\mathrm{Per}(E_f(t,U),U)dt;$$

- the **level total curvature integral** of f

$$\mathrm{LTC}_f(h,U) = \int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_f(t,U),U)dt.$$

Note that $\mathrm{LP}_f(1,U) = \|Df(U)\| = V_f(U)$ by co-area formula and we note $\mathrm{LTC}_f(U) = \mathrm{LTC}_f(1,U)$.

Co-area formula for elementary functions

Proposition : for f an elementary function in U, writing $H(t) = \int_0^t h(s)ds$ we obtain

$$\begin{aligned}
\mathrm{LP}_f(h,U) &= \int_{\mathcal{R}_f} [H(f^+(x)) - H(f^-(x))]\mathcal{H}^1(dx) \\
\mathrm{LTC}_f(h,U) &= \int_{\mathcal{R}_f} [H(f^+(x)) - H(f^-(x))]\kappa_f(x)\mathcal{H}^1(dx) \\
&+ \sum_{x\in\mathcal{C}_f} [H(f^+(x)) - H(f^-(x))]\alpha_f(x) \\
&+ \sum_{x\in\mathcal{I}_f} [H(f^+(x)) + H(f^-(x)) - H(f_-^+(x)) - H(f_+^-(x))]\beta_f(x).
\end{aligned}$$

Proposition : If f, g are elementary functions with $\mathcal{S}_f \cap \mathcal{S}_g$ a finite set of $\{x \in \mathcal{R}_f \cap \mathcal{R}_g; d_S^1(\nu_f(x), \nu_g(x)) \in (0,\pi)\}$, then $f+g$ is an elementary function. Surprisingly, for $h = 1$

$$V_{f+g}(U) = V_f(U) + V_g(U) \text{ and } \mathrm{LTC}_{f+g}(U) = \mathrm{LTC}_f(U) + \mathrm{LTC}_g(U).$$

Application to Shot Noise

Using Slivnyak-Mecke formula, Fubini and stationarity

$$\mathbb{E}(\mathrm{LP}_X(h,U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^+}\int_{\mathcal{R}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X(0)+s))ds\mathcal{H}^1(dx)F(dm)$$

$$\mathbb{E}(\mathrm{LTC}_X(h,U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^+}(A_R(m)+A_C(m)+A_I(m))\,F(dm)$$

$$A_R(m) = \int_{\mathcal{R}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X(0)+s))ds\ \ \kappa_{g_m}(x)\mathcal{H}^1(dx)$$

$$A_C(m) = \sum_{x\in\mathcal{C}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X(0)+s))ds\alpha_{g_m}(x)$$

$$A_I(m) = \frac{\lambda}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))$$

$$\times\int_{g_m^-(z-x)}^{g_m^+(z-x)}\mathbb{E}\left(h(X(0)+s+g_{m'}^+(z))-h(X(0)+s+g_{m'}^-(z))\right)ds\ )\,dxF(dm').$$

Application to Shot Noise

Considering $g = \mathbf{1}_D$ we have $g_m = \mathbf{1}_{D_m}$ so that $\mathcal{R}_{g_m} = \partial D_m$, $\mathcal{C}_{g_m} = \emptyset$, $g_m^-(x) = 0$, $g_m^+(x) = 1$. Hence, writing

$$\overline{p} = \int_{\mathbb{R}^+} \mathcal{H}^1(\partial D_m) F(dm) = 2\pi \int_{\mathbb{R}^+} m F(dm) = \mathbb{E}(2\pi R),$$

$$\begin{aligned}
\mathbb{E}(\mathrm{LP}_X(h, U)) &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^+} \int_{\mathcal{R}_{g_m}} \int_{g_m^-(x)}^{g_m^+(x)} \mathbb{E}(h(X(0)+s)) ds \mathcal{H}^1(dx) F(dm) \\
&= \lambda \mathcal{L}(U) \overline{p} \int_0^1 \mathbb{E}(h(X(0)+s)) ds \\
&= \int_{\mathbb{R}} h(t) (\lambda \mathcal{L}(U) \overline{p} \mathbb{P}(X(0) = [t])) dt \\
&= \int_{\mathbb{R}} h(t) \mathbb{E}(\mathcal{H}^1(\partial E_X(t, U) \cap U)) dt.
\end{aligned}$$

Application to Shot Noise

Moreover, $\mathbb{E}(\mathrm{LTC}_X(h,U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^+}(A_R(m)+A_I(m))\,F(dm)$ with

$$\int_{\mathbb{R}^+}\int_{\partial D_m}\kappa_{g_m}(x)\mathcal{H}^1(dx)F(dm) = \int_{\mathbb{R}^+}\mathrm{TC}(\partial D_m)F(dm) = 2\pi\overline{\chi} = 2\pi, s.t.$$

$$\int_{\mathbb{R}^+}A_R(m)F(dm) = 2\pi\overline{\chi}\int_0^1\mathbb{E}(h(X(0)+s))ds = \int_{\mathbb{R}}h(t)2\pi\overline{\chi}\mathbb{P}(X(0)=[t])dt.$$

According to the kinematic formula we have

$$\int_{\mathbb{R}^2}\sum_{z\in\tau_x\partial D_m\cap\partial D_{m'}} d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))dx = 2\pi m\times 2\pi m' s.t$$

$$\int_{\mathbb{R}^+}A_I(m)F(dm) = \frac{\lambda}{2}\overline{p}^2\int_0^1\mathbb{E}(h(X(0)+s+1)) - \mathbb{E}(h(X(0)+s))ds.$$

It follows that

$$\mathbb{E}(\mathrm{LTC}_X(h,U))$$

$$= \int_{\mathbb{R}}h(t)2\pi\lambda\mathcal{L}(U)\left((\overline{\chi}-\frac{\lambda}{4\pi}\overline{p}^2)\mathbb{P}(X(0)=[t]) + \frac{\lambda}{4\pi}\overline{p}^2\mathbb{P}(X(0)=[t-1])\right)dt.$$

Illustration

We can conclude that for all $k \in \mathbb{N}$ and $t \in (k, k+1]$

$$\mathbb{E}(\mathrm{Per}(E_X(t,U),U)) = \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \times \bar{p}, \text{ and}$$

$$\mathbb{E}(\mathrm{TC}(\partial E_X(t,U),U) = 2\pi \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left(\frac{\bar{p}^2}{4\pi \bar{a}} k + \bar{\chi} - \frac{\bar{p}^2}{4\pi} \lambda \right).$$

Is it far from $2\pi \mathbb{E}(\chi(E_X(t,U)))$?

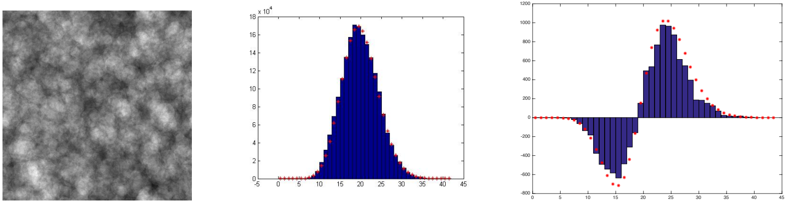
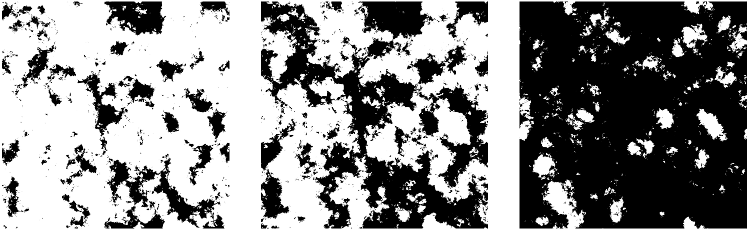


Figure – Shot noise on a domain of size 2000×2000 pixels, with intensity $\lambda = 0.001$, and random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5). Empirical Perimeter and Euler characteristic vs "theoretical" ones

Illustration



$t = 15$ $t = 19$ $t = 25$

Figure – Critical levels

Illustration for squares

If $g = \mathbf{1}_C$ we can also compute and get that for all $k \in \mathbb{N}$ and $t \in (k, k+1]$

$$\mathbb{E}(\mathrm{Per}(E_X(t,U),U)) = \lambda \mathcal{L}(U) e^{-\lambda \overline{a}} \frac{(\lambda \overline{a})^k}{k!} \times \overline{p}, \text{ and}$$

$$\mathbb{E}(\mathrm{TC}(\partial E_X(t,U),U) = 2\pi\lambda\mathcal{L}(U) e^{-\lambda \overline{a}} \frac{(\lambda \overline{a})^k}{k!} \left(\frac{\overline{p}^2}{16\overline{a}} k + \overline{\chi} - \frac{\overline{p}^2}{16}\lambda\right).$$

Is it far from $2\pi\mathbb{E}(\chi(E_X(t,U)))$?

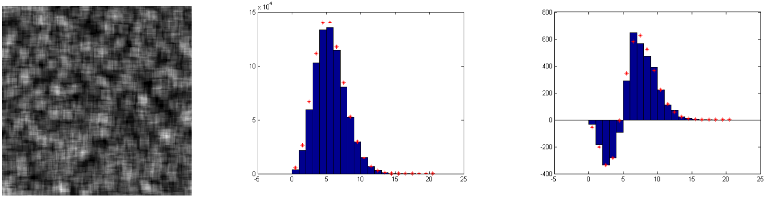
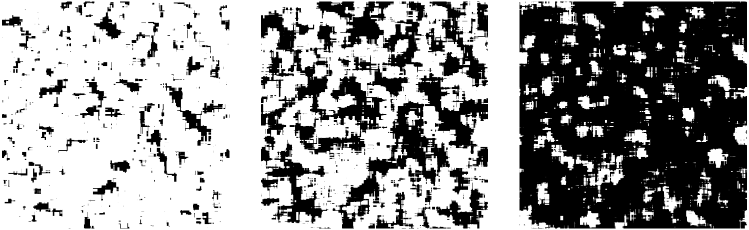


Figure – Shot noise on a domain of size 2000×2000 pixels, with intensity $\lambda = 0.005$, and random square of side length $R = 100$. Empirical Perimeter and Euler characteristic vs "theoretical" ones

Illustration



$t = 3$ $t = 5$ $t = 8$

Figure – Critical levels

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