Probabilistic properties of non-conventional ergodic averages

Evgeny Verbitskiy

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Ergodic averages

Measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$, observable $f : X \to \mathbb{R}$

$$S_N(x) = S_N^f(x) = S_N f(x) := \sum_{k=0}^{N-1} f(T^k x).$$

Ergodic theorems establish convergence of

$$\frac{1}{N} S_n^f(x)$$

in norm (L_2, von Neumann) and pointwise (Birkhoff).

Furstenberg's non-conventional multiple egrodic averages

Observables f_m, $m = 1, \ldots, \ell$,

$$A_n(x) = A_N^{f_1,\ldots,f_\ell}(x) := \sum_{k=0}^{N-1} f_1(T^k x) f_2(T^{2k} x) \ldots f_\ell(T^{\ell k} x).$$

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Furstenberg's multiple recurrence: $\mu(A) > 0$

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu\left(\left\{x :\ T^k x, T^{2k} x, \ldots, T^{\ell k} x \in A\right\}\right) > 0.$$

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Polynomial version (Bergelson, Leibman)

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu\left(\left\{x :\ T^{p_1(k)} x, T^{2k} x, \ldots, T^{p_\ell(k)} \in A\right\}\right) > 0.$$

Convergence

Furstenberg

If $T :\rightarrow X$ is **weakly mixing**, then

$$\frac{1}{N}\sum_{k=1}^{N}\prod_{j=1}^{\ell} f_j(T^{jk}x) \rightarrow \prod_{j=1}^{\ell}\int f_j d\mu \quad \text{in } L^2(\mu).$$

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For a general ergodic system Fursteberg's averages **need not** converge to a constant function

Bergelson's PET

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Convergence

$$\frac{1}{N}A_N = \frac{1}{N}\sum_{k=1}^{N} f_1 \circ T^k f_2 \circ T^{2k} \dots f_\ell \circ T^{\ell k}$$

Norm convergence: general measure preserving T

$\ell = 2$ (Furstenberg), $\ell = 3$ (Furstenberg-Weiss, Host-Kra), $\ell = 4$ (Ziegler), $\ell \in \mathbb{N}$ (Host-Kra, Ziegler)

Almost sure convergence

- Bourgain's double ET (1990): T_1, T_2 are powers of ergodic T, $N^{-1}\sum_{k=1}^{N} f_1 \circ T_1^n f_2 \circ T_2^n$ converge a.s.
- Assani (1998): weakly mixing T+smth extra, then

$$N^{-1}\sum_{k=1}^{N} f_1 \circ T^k f_2 \circ T^{2k} \dots f_\ell \circ T^{\ell k} \to \prod_{j=1}^{\ell} \int f_j d\mu$$

Objective

Finer probabilistic properties,
i.e., beyond convergence

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- **Central Limit Theorems**
- **Large Deviations**
- **Thermodynamic Formalism**
- **Multifractal Analysis**

Chapter II.

Central Limit Theorems

CLT

(1) If $\{X_k\}$ is a stationary sequence of **weakly dependent** random variables with finite moments $\mu = \mathbb{E}X_k$, $\mathbb{E}X^p < \infty, p \geq 2$, then

$$\frac{X_0 + \ldots + X_{n-1} - n\mu}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2).$$

where

$$\sigma^2 = \mathrm{var}(X_0) + 2\sum_{k=1}^{\infty} \mathrm{cov}(X_0, X_k) < \infty.$$

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(2) "Hyperbolic" dynamical systems: If $T : X \to X$ is **rapidly mixing**, and $f : X \to \mathbb{R}$ is **smooth**, then

$$X_n = f(T^n x), \quad n = 0, 1, \ldots.$$

satisfies the CLT.

CLT

(3) For an arbitrary **aperiodic** m.p.d.s. (X, μ, T), there exists $f \in L^2(\mu)$ with $\int f d\mu = 0$ such that

$$\frac{f(\omega) + \ldots + f(T^{n-1}\omega)}{||(S_n f)||_2} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

[Burton & Denker, Lacey (T_α)]

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(4) Raikov - Riesz - Kac:

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x)$$

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Lacunary series (Salem, Zygmund,...)

$$m\left(\left\{x \in [0,1] : \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos(2\pi n_k x) \leq z\right\}\right) \to \Phi_{0,\frac{1}{2}}(z).$$

CLT for non-conventional averages

(5) generalized Riesz-Raikov sums: under minor technical assumptions of $f_1, \ldots, f_\ell$,

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} f_1(\theta^{p_1(n)}x) \ldots f_\ell(\theta^{p_\ell(n)}x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

where $p_1, \ldots, p_\ell$ are polynomials: $p_k(n) - p_m(n) \to \infty$ as $n \to \infty$. [K. Fukuyama (2000)]

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(6) Irrational circle rotations (Weber, 2006):

$$S_N^T(f,g)(t) = \sum_{n=1}^{N} f(T^n t) g(T^{2n} t), \qquad \frac{S_N^T(f,g)}{||S_N^T(f,g)||_2} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where

$$f = \sum_{m=1}^{\infty} a_m \cos(2\pi l_m t), \quad g = \sum_{m=1}^{\infty} b_m \cos(2\pi l_m t),$$

under some conditions on $\{a_m, b_m, l_m\}$.

CLT for non-conventional averages

(7) Kifer (2010): CLT for sums

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}[F(X_0(n), X_1(q_1(n)), X_2(q_2(n)), \ldots, X_\ell(q_\ell(n))) - \overline{F}]$$

(1) X_i's are bounded, exponentially fast ψ-mixing random variables with some stationarity properties;

(2) F is Lipschitz;

(3) $\overline{F} = \int F\, d\Big(\mu_0 \times \mu_1 \times \cdots \times \mu_\ell\Big)$, where $X_j(0) \sim \mu_j$;

(4) q_i's are integer valued on integers +a growth conditions

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- $X_i(n) = f_i(\xi_n)$, $\{\xi_n\}$ is a Markov chain satisfying Doeblin's condition, μ being the invariant measure.
- $X_i(n) = f_i \circ T^n$, T is **hyperbolic**, μ is invar. Gibbs.

(8) Kifer & Varadhan (2014): cont. time., ...

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Proving CLT's

- Characteristic function method

$$\phi_n(t) = \mathbb{E}e^{itY_n} \to \mathbb{E}e^{itY} = \phi(t) \Rightarrow Y_n \xrightarrow{\mathcal{D}} Y.$$

- Bernstein's block method
- Martingale method

Strong dependence between **past** and **future** terms in

$$\sum_{n=1}^{N} f_1(X(n)) f_2(X(2n))$$

K. (2010): martingale techniques do not seem to work
K. & V. (2014): appropriately modified martingale approach works

★1

Chapter III.

Large deviations

Basic setup

Suppose $\{S_n\}$ is a sequence of random variables

$$\frac{1}{n}S_n \to \text{const} \quad \mathbb{P} - a.s.$$

$$\left(\text{e.g., } S_n = \sum_{k=1}^{n} f(T^k x),\ S_n = \sum_{k=1}^{n} f(T^k x) g(T^{2k} x), \ldots, \right),$$

and we would like to understand

$$\mathbb{P}\left(\frac{1}{n}S_n \in C\right), \quad C \subset \mathbb{R}.$$

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Logarithmic moment generating function/ free energy

$$\Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}} \exp(tS_n).$$

Logarithmic moment generating function

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Theorem (Gärtner-Ellis).

If $\Lambda(t)$ exists and is finite for all t, then introducing

$$\Lambda^*(x) = \sup_t \Big(tx - \Lambda(t) \Big),$$

one has

- $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} S_n \in F\right) \le -\inf_{x\in F} \Lambda^*(x); \forall F$ clsd.
- $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} S_n \in O\right) \ge -\inf_{x\in O} \Lambda^*(x); \forall O$ open

Examples

(1) Full shift $X = \mathcal{A}^{\mathbb{Z}_+}$, continuous observable $f : X \to \mathbb{R}$ and a Bowen-Gibbs measure $\mathbb{P}$ for potential ϕ:

$$\frac{1}{C} \leq \frac{\mathbb{P}([x_0, \ldots, x_{n-1}])}{\exp\left((S_n\phi)(x) - nP\right)} \leq C.$$

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Then

$$\int e^{t(S_n f)(x)} \mathbb{P}(dx) \asymp \sum_{[x_0^{n-1}]} e^{tS_n f(x^*)} \times e^{(S_n\phi)(x) - nP}$$

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$$\Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{[x_0^{n-1}]} e^{tS_n f(x^*) + (S_n\phi)(x) - nP} = P(\phi + tf) - P(\phi).$$

Examples

Carinci et al 2012

(2) Full shift $X = \{-1, 1\}^{\mathbb{N}}$, $\mathbb{P} = \mathsf{Ber}(p)$. Observable

$$A_n = \sum_{k=1}^{n} x_k x_{2k} \quad \left(A_n = \sum_{k=1}^{n} x_k x_{2k} \ldots x_{\ell k} \right)$$

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Can we compute

$$\Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{tA_n} = \Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp \left(t \sum_{k=1}^{n} x_k x_{2k} \right)$$

Multiplicative decomposition

$$\begin{aligned}\sum_{k=1}^{n} x_k x_{2k} &= \Big(x_1 x_2 + x_2 x_4 + \dots\Big) + \\ &\quad \Big(x_3 x_6 + x_6 x_{12} + \dots\Big) + \\ &\quad \Big(x_5 x_{10} + x_{10} x_{20} + \dots\Big) \\ &= \sum_{j \text{ odd}} \sum_{m=1}^{M(n,j)} x_{2^{m-1}j} x_{2^m j} = \sum_{j \text{ odd}} \sum_{m=1}^{M(n,j)} \tau_{m-1}^{(j)} \tau_m^{(j)} \\ &= \sum_{j \text{ odd}} S_j^n(\tau_0^{(j)}, \tau_1^{(j)}, \dots).\end{aligned}$$

Multiplicative decomposition, II

Notation:

$$Z_k(t) = \mathbb{E} \exp \left(t \sum_{m=1}^{k} \tau_{m-1} \tau_m \right), \quad \{\tau_m\} \text{ Bernoulli}$$

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$$\mathbb{E}\exp\left(t\sum_{k=1}^{n}x_k x_{2k}\right) = \prod_{j\text{ odd}} Z_{k(n,j)}(t), \quad k(n,j) = \left\lfloor \log_2 \frac{n}{j} \right\rfloor.$$

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$$\Lambda(t) = \sum_{m=1}^{\infty} \frac{1}{2^{m+1}} G_m, \qquad G_m \quad \textbf{explicit}.$$

$$\frac{\#\{j : \quad k(n,j) = m\}}{n} \to \frac{1}{2^{m+1}}$$

Same method works for

$$S_N^{(q)} = \sum_{k=1}^{N} X_k X_{2k} X_{3k} \ldots X_{qk}, \qquad X_k \sim \mathsf{Ber}(p).$$

More general: Kifer, Varadhan (2014).

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Question

What could be said about LD's of

$$S_N^{(3)} = \sum_{k=1}^{N} X_k X_{2k} X_{3k} \dots X_{qk}$$

where (X_k) is Markov with values in $\mathcal{A}$, e.g., $\mathcal{A} = \{-1, 1\}$, under the translation invariant Markov measure.

Large deviations

(2) Kifer, Varadhan (2014)

- **nice** Markov chain $(X(0), X(1), X(2), \ldots)$ with values in Polish $(M, \mathcal{B})$.
- **bounded measurable** observable $W(x_1, \ldots, x_\ell)$.
- **linear** $q_j(n) = jn$ for $j = 1, \ldots, k$
- **faster growth** $q_j(n)$, $j = k+1, \ldots, \ell$.

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- **faster growth** $q_j(n)$, $j = k+1, \ldots, \ell$.

Theorem. The following limit exists and is independent of $x \in M$

$$\Lambda(t) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_x \Big(t \sum_{n=1}^{N} W(X(q_1(n)), \ldots X(q_\ell(n)) \Big).$$

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- X is a SFT, $T : X \to X$ left shift, μ is a (Bowen-) Gibbs measure for Hölder continuous g.
- $g_1(n) = n$, but $g_2(n)$,..., $g_\ell(n)$ **grow faster**.
- $\Phi : X^\ell \to \mathbb{R}$ is continuous

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Then the following limit exists

$$\Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \int \exp\Big(t \sum_{k=1}^{n} \Phi(T^{g_1(k)}x, \ldots, T^{g_\ell(k)}x)\Big)\mu(dx)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \int \exp\Big(\sum_{k=1}^{n} \widehat{\Phi}_t(T^k x)\Big)\mu(dx) = P(\widehat{\Phi}_t + g) - P(g)$$

$$\widehat{\Phi}_t(x) = \log \int \exp\Big(t\Phi(x, z_2, \ldots, z_\ell)\Big)\mu(dz_2)\cdots\mu(dz_\ell).$$

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Chapter IV.

Thermodynamic Formalism

Gibbs measures

DLR Gibbs measures

$$\mathbb{P}(x_{[1:n]}|x_{[1:n]^c}) = \frac{1}{Z_n}\exp\Big(-H_n(x)\Big), \quad \text{e.g.,} \quad H = J\sum_{i\sim j} x_i x_j .$$

Bowen Gibbs measures

$$\frac{1}{c} \leq \frac{\mathbb{P}(x_{[1:n]})}{\exp\Big((S_n\phi)(x) - nP\Big)} \leq c$$

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Morally, $H_n(x) \approx (S_n\phi)(x)$,
but definitions are not equivalent

Gibbs measures for multiplicative potentials

Chazottes, Redig

Can we define Gibbs measure for Hamiltonians like

$$H = -J \sum x_j x_{2j} - h \sum x_j,$$

or more generally, for

$$H = \sum_A U(A, x_A), \quad U(qA, x) = U(A, T_q x), \quad (T_q x)_m = x_{qm}$$

★$_2$

Bowen Gibbs measures for non-conventional potentials

Definition. We say that a probability measure $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ is called Bowen-Gibbs for a non-conventional potential $\phi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ if there exists a constant P and $c > 1$ such that

$$\frac{1}{c} \leq \frac{\mathbb{P}(x_{[1:2n]})}{\exp\left(\sum_{j=1}^{n} \phi(x_j, x_{2j}) - nP\right)} \leq c$$

for every $n \geq 1$ and all x.

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- Not clear this definition makes sense.
- Fan-Schmeling-Wu measure comes close, but not Bowen-Gibbs in the above sense.

Chapter V.

Multifractal Analysis

Multifractal analysis of ergodic averages

$T : \Sigma \to \Sigma$, continuos observable $f : \Sigma \to \mathbb{R}$.

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Multifactal decomposition

$$\Sigma = \bigcup_{\alpha \in \mathbb{R}} K_\alpha \bigcup K_\sharp,$$

$$K_\alpha = \Big\{ x \in \Sigma : \lim_{n\to\infty} \frac{1}{n}(S_n\phi)(x) = \alpha \Big\},$$

$$K_\sharp = \Big\{ x \in \Sigma : \lim_{n\to\infty} \frac{1}{n}(S_n\phi)(x) \text{ does not exist} \Big\}.$$

Multifractal analysis of ergodic averages

$T:\Sigma\to\Sigma$, continuos observable $f:\Sigma\to\mathbb{R}$.

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$$\Sigma=\bigcup_{\alpha\in\mathbb{R}}K_\alpha\bigcup K_\sharp,$$
$$K_\alpha=\Big\{x\in\Sigma:\ \lim_{n\to\infty}\frac{1}{n}(S_n\phi)(x)=\alpha\Big\},$$
$$K_\sharp=\Big\{x\in\Sigma:\ \lim_{n\to\infty}\frac{1}{n}(S_n\phi)(x)\text{ does not exist}\Big\}.$$

Theorem.

For all α with $K_\alpha\neq\varnothing$, one has

$$h_{\text{top}}(K_\alpha)=\sup\Big\{h_\mu(f):\ \mu\text{ is invariant and }\mathbb{E}_\mu\phi=\alpha\Big\}$$
$$=P_\phi^*(\alpha)=\inf_{s\in\mathbb{R}}\Big(-s\alpha+P(s\phi)\Big).$$

Multifractal analysis of Furstenberg averages

Fan, Liao, and Ma (2011)

Consider $\Sigma = \{-1, 1\}^{\mathbb{N}}$. For $\theta \in [-1, 1]$, let

$$B_\theta := \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

Then

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\left(\frac{1+\theta}{2}\right).$$

Multifractal analysis of Furstenberg averages

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Kifer (2012)

Let r be prob. meas. on $\mathcal{A} = \{0, \ldots, m-1\}$, then the set

$$\left\{ (x_k) : \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}[x_k = a_1, x_{q_2(k)=a_2}, \ldots x_{q_\ell(k)=a_\ell}] \to \prod_{j=1}^{\ell} r_{a_j} \ \forall a_1^\ell \right\}$$

has Hausdorff dimension

$$\frac{-\sum_{j=0}^{m-1} r_j \log r_j}{\log m}$$

On symbolic space $\Sigma = \{0,1\}^{\mathbb{N}}$, consider the level sets

$$A_\alpha := \left\{(x_k)_1^\infty \in \Sigma : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k x_{2k} = \alpha\right\} \qquad (\alpha \in [0,1]).$$

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What about the multiplicative golden subshift

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- Kenyon, Peres and Solomyak

  $$\dim_H(A_0) = \dim_H(A_0') = -\log(1-p), \quad p^2 = (1-p)^3.$$

- Peres-Solomyak '12; Fan, Schmeling and Wu '11-'16:

  $$\dim_H A_\alpha = -\log_2(1-p) - \frac{\alpha}{2} \log_2 \frac{q(1-p)}{p(1-q)},$$

  $$\text{where } \begin{cases} p^2(1-q) = (1-p)^3, \\ 2pq = \alpha(2+p-q). \end{cases}$$

Theorem (Fan,Schmeling, Wu)

Let $\Sigma_m = \{0, \ldots, m-1\}^{\mathbb{N}}$ *and* $\phi : \mathcal{A}^\ell \to \mathbb{R}$.
For any $\alpha \in \mathbb{R}$, *let*

$$E_\alpha = \left\{ x \in \Sigma_m : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(x_k, x_{kq}, \ldots, x_{kq^{l-1}}) = \alpha \right\}.$$

Then there exist $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ *such that* $E_\alpha = \varnothing$ *if* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$.
For $\alpha \in (\underline{\alpha}, \overline{\alpha})$, $E_\alpha \neq \varnothing$ *and*

$$\dim_H(E_\alpha) = \frac{P_\varphi^*(\alpha)}{q^{\ell-1}\log m},$$

where

$$P_\varphi^*(\alpha) = \inf_{s\in\mathbb{R}}(-s\alpha + P_\varphi(s))$$

is the Legendre transform of a certain pressure function $P_\phi(\cdot)$ *associated to a non-linear transfer operator.*

Corollary (Variational principle)

Under the conditions of the previous Theorem, for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, $E_\alpha \neq \varnothing$ *and*

$$\dim_H(E_\alpha) = \frac{(q-1)^2}{q^{l-1}\log m} \sup_{\mu \in \mathcal{M}(\Sigma_m, \alpha)} \left\{ \sum_{k=1}^{\infty} \frac{H_k(\mu)}{q^{k+1}} \right\},$$

where $H_k(\mu) = -\sum_{x_1^k} \mu(x_1^k) \log \mu(x_1^k)$ *and the supremum is taken over all probability measures on* Σ_m *such that*

$$(q-1)^2 \sum_{k=1}^{\infty} \frac{\mathbb{E}_\mu(S_k\phi)}{q^{k+1}} = \alpha$$

$$\dim_H(E_\alpha) = \frac{1}{\log m} \sup\left\{ h_\mu(f) : \mu \text{ is invariant, } \mathbb{E}_\mu \phi = \alpha \right\}$$