Impact of a sample procedure on a Gibbs tree

Stéphane Seuret, Université Paris-Est Créteil

Workshop on Probabilistic Aspects of Multiple Ergodic Averages

joint work with J. Barral

1 - Introduction: What is my question?

Notations: $\Sigma_j = \{0,1\}^j$ is the set of finite words of length j, and $\Sigma = \{0,1\}^{\mathbb{N}}$.
The word $w \in \Sigma_j$ is written $w = (w_1, w_2, ..., w_j)$, and $|w| = j$.

1 - Introduction: What is my question?

Notations: $\Sigma_j = \{0,1\}^j$ is the set of finite words of length j, and $\Sigma = \{0,1\}^{\mathbb{N}}$.

The word $w \in \Sigma_j$ is written $w = (w_1, w_2, ..., w_j)$, and $|w| = j$.

For $w \in \Sigma_j$, the dyadic interval I_w is $I_w = \left[x_w := \sum_{k=1}^{j} w_k 2^{-k}, x_w + 2^{-j} \right)$.

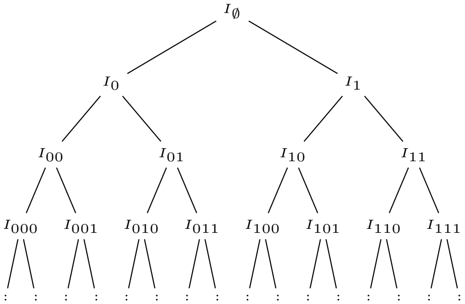
For $x \in [0,1]$, $I_j(x) = I_w$ where w is the unique word j such that $x \in I_w$.

1 - Introduction: What is my question?

Notations: $\Sigma_j = \{0,1\}^j$ is the set of finite words of length j, and $\Sigma = \{0,1\}^{\mathbb{N}}$.
The word $w \in \Sigma_j$ is written $w = (w_1, w_2, ..., w_j)$, and $|w| = j$.

For $w \in \Sigma_j$, the dyadic interval I_w is $I_w = \left[x_w := \sum_{k=1}^{j} w_k 2^{-k}, x_w + 2^{-j}\right)$.

For $x \in [0,1]$, $I_j(x) = I_w$ where w is the unique word j such that $x \in I_w$.

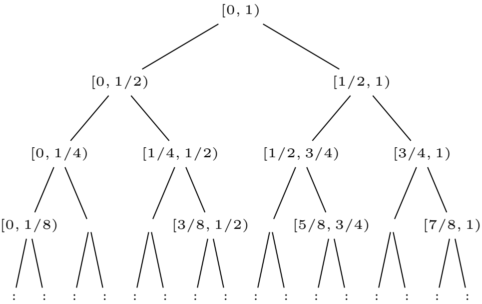


1 - Introduction: What is my question?

Notations: $\Sigma_j = \{0,1\}^j$ is the set of finite words of length j, and $\Sigma = \{0,1\}^{\mathbb{N}}$.
The word $w \in \Sigma_j$ is written $w = (w_1, w_2, ..., w_j)$, and $|w| = j$.

For $w \in \Sigma_j$, the dyadic interval I_w is $I_w = \Big[x_w := \sum_{k=1}^{j} w_k 2^{-k}, x_w + 2^{-j} \Big)$.

For $x \in [0,1]$, $I_j(x) = I_w$ where w is the unique word j such that $x \in I_w$.

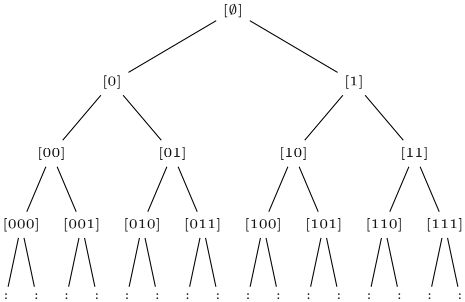


1 - Introduction: What is my question?

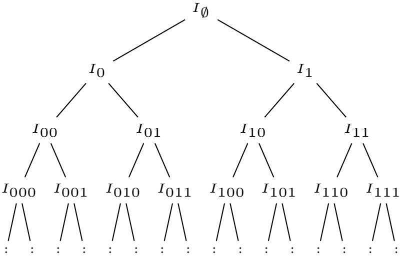
Notations: $\Sigma_j = \{0,1\}^j$ is the set of finite words of length j, and $\Sigma = \{0,1\}^{\mathbb{N}}$.
The word $w \in \Sigma_j$ is written $w = (w_1, w_2, ..., w_j)$, and $|w| = j$.

For $w \in \Sigma_j$, the dyadic interval I_w is $I_w = \left[x_w := \sum_{k=1}^{j} w_k 2^{-k}, x_w + 2^{-j} \right)$.

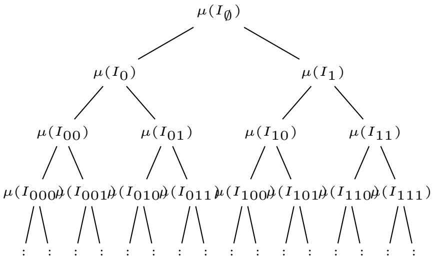
For $x \in [0,1]$, $I_j(x) = I_w$ where w is the unique word j such that $x \in I_w$.



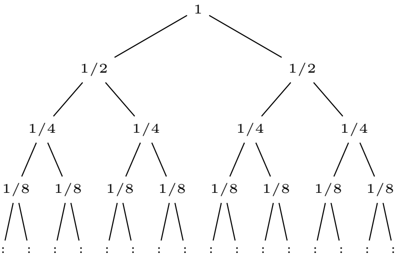
Take any measure μ on $[0, 1]$, and build the associated dyadic tree:



Take any measure μ on $[0,1]$, and build the associated dyadic tree:

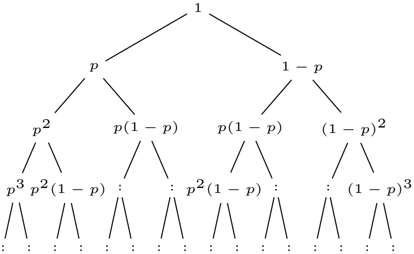


Take any measure μ on $[0, 1]$, and build the associated dyadic tree:



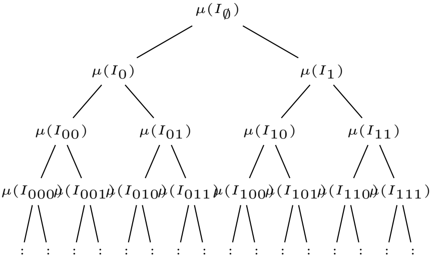
$\mu =$ Lebesgue measure on $[0, 1]$

Take any measure μ on $[0, 1]$, and build the associated dyadic tree:

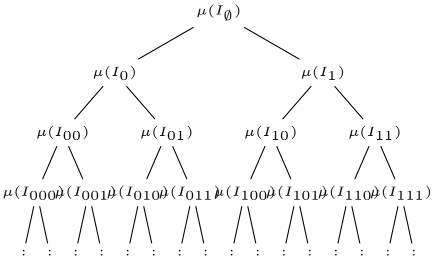


μ = Binomial measure with parameter $p \in (0, 1)$

Take any measure μ on $[0,1]$, and build the associated dyadic tree:



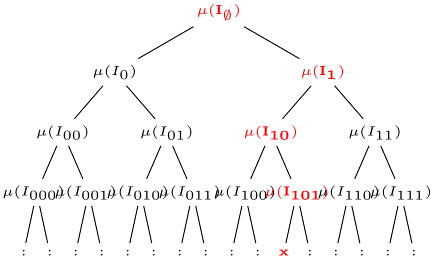
Take any measure μ on $[0,1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j\to+\infty} \frac{\log_2 \mu(I_j(x))}{-j}$$

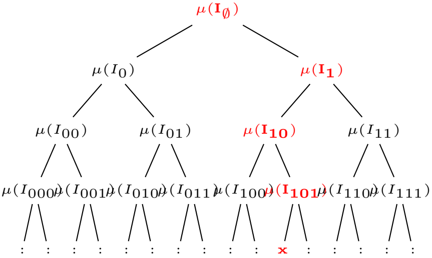
Take any measure μ on $[0,1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j \to +\infty} \frac{\log_2 \mu(I_j(x))}{-j}$$

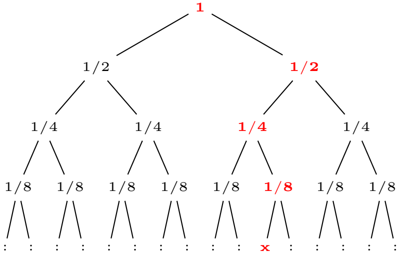
Take any measure μ on $[0,1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j \to +\infty} \frac{\log_2 \mu(I_j(x))}{-j} \;\to\; \textbf{Idea:}\; \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

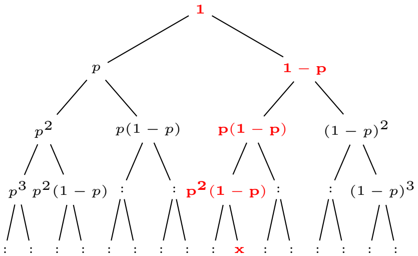
Take any measure μ on $[0,1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j \to +\infty} \frac{\log_2 \mu(I_j(x))}{-j} \ \rightarrow \ \textbf{Idea:} \ \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

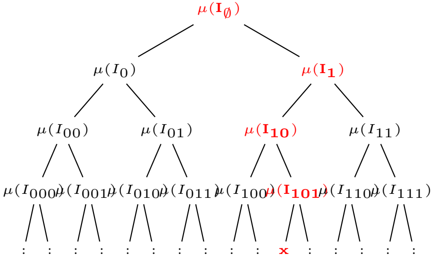
Take any measure μ on $[0, 1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0, 1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j \to +\infty} \frac{\log_2 \mu(I_j(x))}{-j} \;\to\; \textbf{Idea: } \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

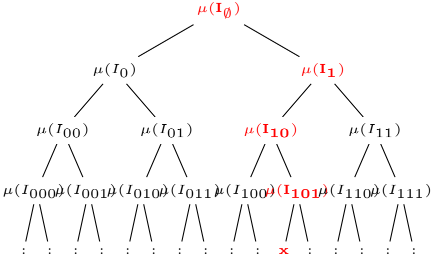
Take any measure μ on $[0,1]$, and build the associated dyadic tree:



Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu,x) := \liminf_{j\to+\infty} \frac{\log_2 \mu(I_j(x))}{-j} \;\rightarrow\; \textbf{Idea: } \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

Take any measure μ on $[0,1]$, and build the associated dyadic tree:



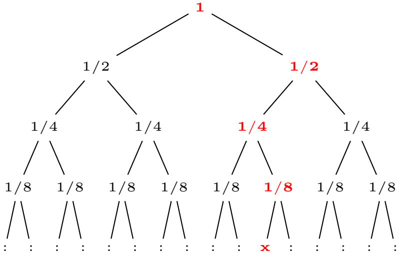
Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu,x) := \liminf_{j\to+\infty} \frac{\log_2 \mu(I_j(x))}{-j} \;\rightarrow\; \textbf{Idea: } \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

One tries to "understand" the level sets
$E_\mu(h) = \{x \in [0,1] : \underline{\dim}(\mu,x) = h\}$

and to compute the **multifractal spectrum**
$D_\mu(h) = \dim E_\mu(h)$.

Take any measure μ on $[0,1]$, and build the associated dyadic tree:

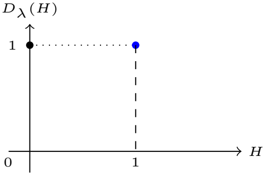


Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

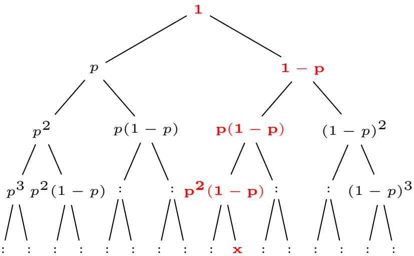
$$\underline{\dim}(\mu,x) := \liminf_{j\to+\infty} \frac{\log_2 \mu(I_j(x))}{-j} \;\rightarrow\; \textbf{Idea: } \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

One tries to "understand" the level sets
$E_\mu(h) = \{x \in [0,1] : \underline{\dim}(\mu,x) = h\}$

and to compute the **multifractal spectrum**
$D_\mu(h) = \dim E_\mu(h)$.



Take any measure μ on $[0,1]$, and build the associated dyadic tree:

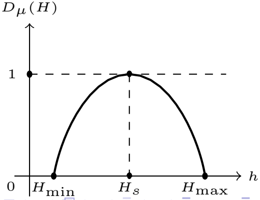


Multifractal analysis: Study of the local dimensions of μ at $x \in [0,1]$:

$$\underline{\dim}(\mu, x) := \liminf_{j\to+\infty} \frac{\log_2 \mu(I_j(x))}{-j} \quad \rightarrow \quad \textbf{Idea: } \mu(I_j(x)) \sim |I_j(x)|^{\underline{\dim}(\mu,x)} = 2^{-j\underline{\dim}(\mu,x)}$$

One tries to "understand" the level sets
$E_\mu(h) = \{x \in [0,1] : \underline{\dim}(\mu, x) = h\}$

and to compute the **multifractal spectrum**
$D_\mu(h) = \dim E_\mu(h)$.

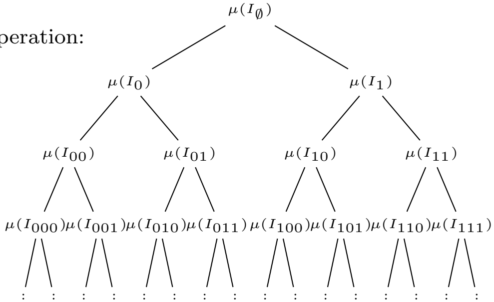


We perform a random sampling on the "measured" tree.

We perform a random sampling on the "measured" tree.

Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:



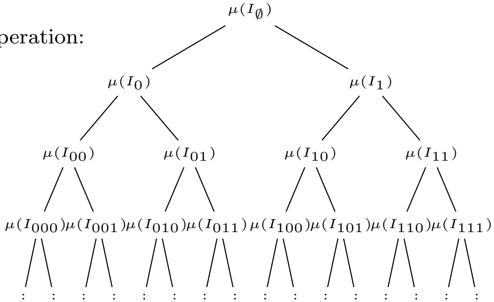
We perform a random sampling on the "measured" tree.

Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.



We perform a random sampling on the "measured" tree.

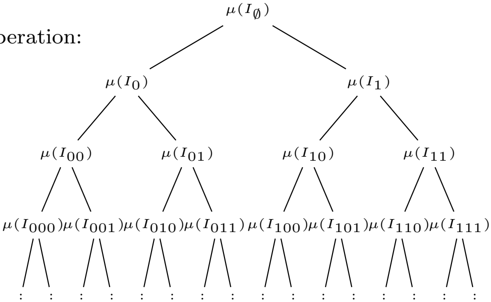
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.



We perform a random sampling on the "measured" tree.

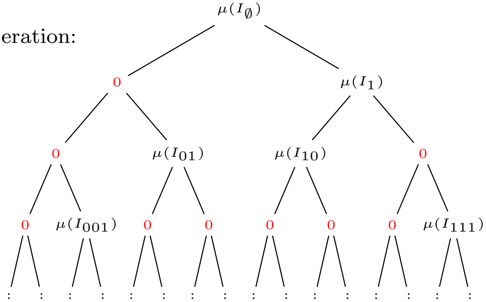
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.



At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial vertices.

We perform a random sampling on the "measured" tree.

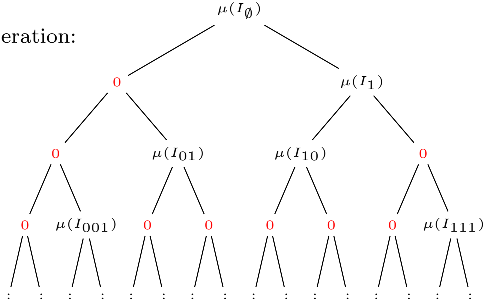
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.



At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial vertices.

Questions:

- Which vertices survive after sampling?
- Can one recover the initial tree?
- What about the structure of the survivors?

We perform a random sampling on the "measured" tree.

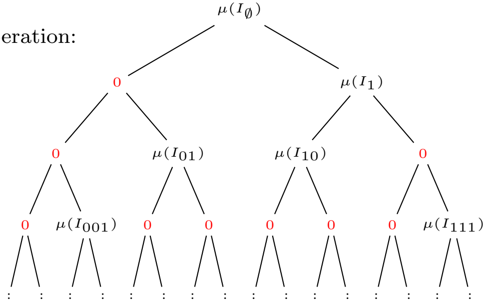
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.



At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial vertices.

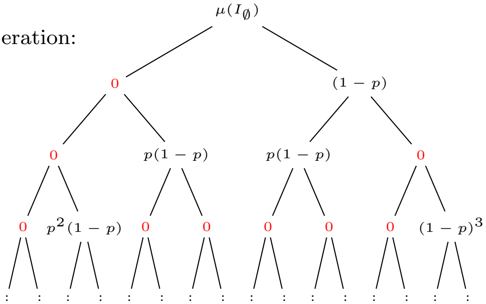
Motivations:

- Natural question, not so far from percolation theory.
- Recovering from sparse data.
- Random wavelet series.

We perform a random sampling on the "measured" tree.

Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:



For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.

At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial vertices.

Questions:

- Which vertices survive after sampling?
- Can one recover the initial tree?
- What about the structure of the survivors?

We perform a random sampling on the "measured" tree.

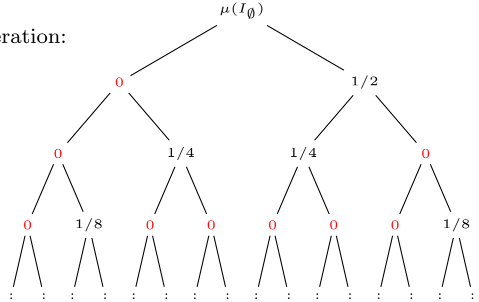
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.



At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial coefficients.

Questions:

- Which vertices survive after sampling?
- Can one recover the initial tree?
- What about the structure of the survivors?

We perform a random sampling on the "measured" treee.

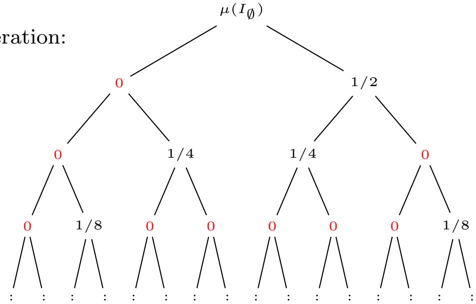
Fix a sampling index $0 < \eta < 1$.

Apply the following (random) operation:

For each $w \in \Sigma_j$, let p_w a Bernoulli r.v. $B(2^{-j(1-\eta)})$.

If $p_w = 1$, keep the value $\mu(I_w)$, and one says that w survives.

If $p_w = 0$, replace it by the value 0 at the vertex w.

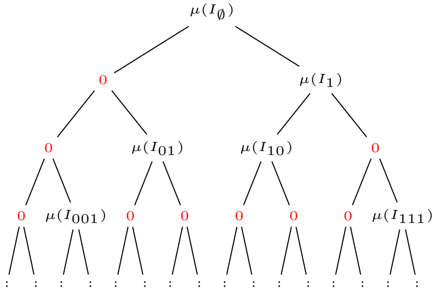


At each generation j, there are $\sim 2^{j\eta}$ survivors amongst the 2^j initial coefficients.

Questions:

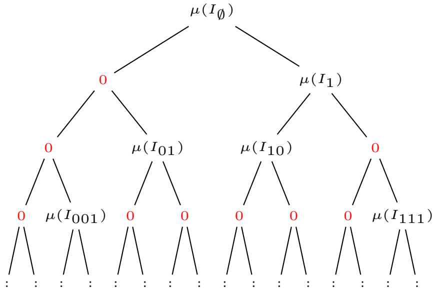
- Which vertices survive after sampling? → Description of the survivors.
- Can one recover the initial tree? → "yes" when $\eta > 1/2$ and μ Gibbs.
- What about the structure of the survivors? → new multifractal behavior(s).

We do not study directly the sampled tree, but an equivalent object.



We do not study directly the sampled tree, but an equivalent object.

Call $\tilde{\mu}$ the new structure.



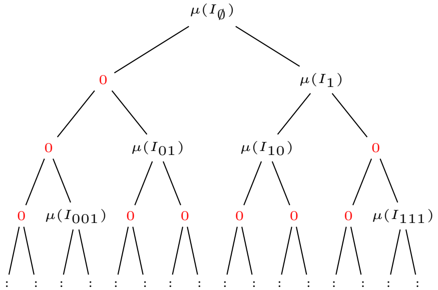
We do not study directly the sampled tree, but an equivalent object.

Call $\tilde{\mu}$ the new structure.

Recalling that the local dimension is

$$\underline{\dim}(\tilde{\mu}, x) = \liminf_{j\to+\infty} \frac{\log_2 \tilde{\mu}(I_j(x))}{-j} \quad ,$$

then, since there are
only few survivors,
this quantity is not relevant
since $\tilde{\mu}(I_j(x)) = 0$ very often.



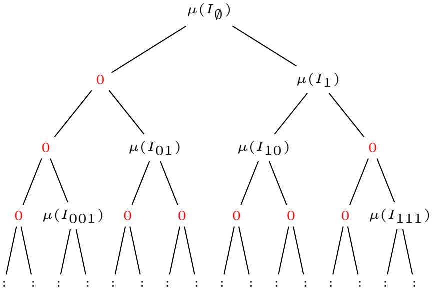
We do not study directly the sampled tree, but an equivalent object.

Call $\tilde{\mu}$ the new structure.

Recalling that the local dimension is

$$\underline{\dim}(\tilde{\mu}, x) = \liminf_{j\to+\infty} \frac{\log_2 \tilde{\mu}(I_j(x))}{-j},$$

then, since there are only few survivors, this quantity is not relevant since $\tilde{\mu}(I_j(x)) = 0$ very often.



We build from $\tilde{\mu}$ a random capacity M_μ.

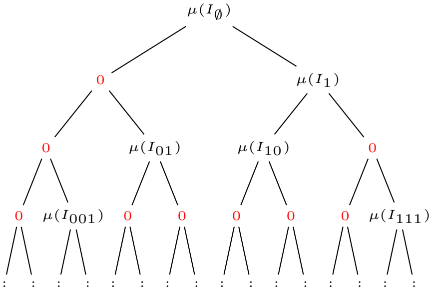
We do not study directly the sampled tree, but an equivalent object.

Call $\tilde{\mu}$ the new structure.

Recalling that the local dimension is

$$\underline{\dim}(\tilde{\mu}, x) = \liminf_{j \to +\infty} \frac{\log_2 \tilde{\mu}(I_j(x))}{-j} ,$$

then, since there are
only few survivors,
this quantity is not relevant
since $\tilde{\mu}(I_j(x)) = 0$ very often.



We build from $\tilde{\mu}$ a random capacity M_μ.
Observe that for any measure μ,

$$\mu(I_W) = \sup\{\mu(I_w) : I_w \subset I_W\}.$$

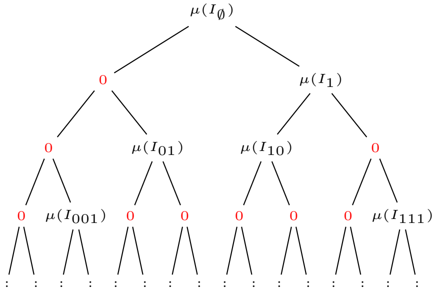
We do not study directly the sampled tree, but an equivalent object.

Call $\tilde{\mu}$ the new structure.

Recalling that the local dimension is

$$\underline{\dim}(\tilde{\mu}, x) = \liminf_{j \to +\infty} \frac{\log_2 \tilde{\mu}(I_j(x))}{-j},$$

then, since there are only few survivors, this quantity is not relevant since $\tilde{\mu}(I_j(x)) = 0$ very often.



We build from $\tilde{\mu}$ a random capacity M_μ.

Observe that for any measure μ,

$$\mu(I_W) = \sup\{\mu(I_w) : I_w \subset I_W\}.$$

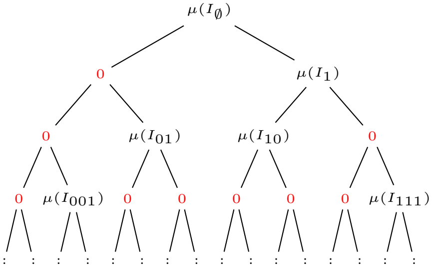
Definition

For every word W, set: $\mathsf{M}_\mu(I_W) := \sup\left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.

Clearly M_μ is a capacity: if $I_w \subset I_W$, $\mathsf{M}_\mu(I_w) \leq \mathsf{M}_\mu(I_W)$.

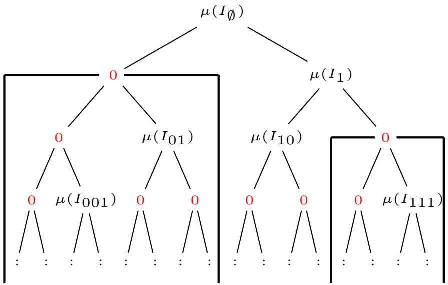
Definition

For every word W, set: $\mathsf{M}_\mu(I_W) := \sup\left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.



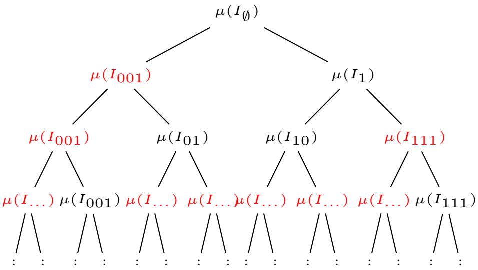
Definition

For every word W, set: $\mathsf{M}_\mu(I_W) := \sup\left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.



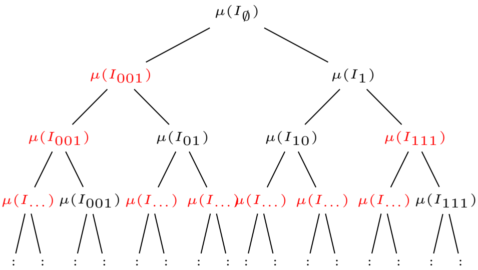
Definition

For every word W, set: $\mathsf{M}_\mu(I_W) := \sup\left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.



Definition

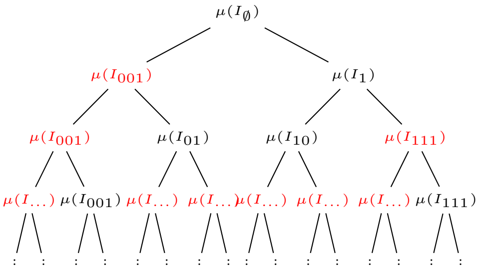
For every word W, set: $\mathsf{M}_\mu(I_W) := \sup\left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.



- "Equivalence" between M_μ and $\tilde{\mu}$.
- If μ has full support in $[0,1]$, the sup is a max (a.s, for every w).
- One always has $\mathsf{M}_\mu(I_w) \leq \mu(I_w)$.
- When w survives, $\mathsf{M}_\mu(I_w) = \mu(I_w)$.
- M_μ combines dynamics and randomness $\longrightarrow$ "phase transitions".

Definition

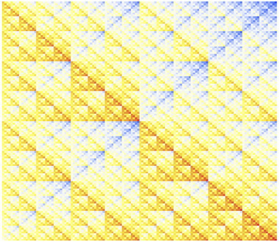
For every word W, set: $\mathsf{M}_\mu(I_W) := \sup \left\{\mu(I_w) : I_w \subset I_W \text{ and } w \text{ survives}\right\}$.

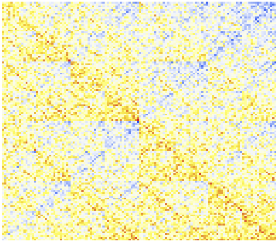


- "Equivalence" between M_μ and $\tilde{\mu}$.
- If μ has full support in $[0,1]$, the sup is a max (a.s, for every w).
- One always has $\mathsf{M}_\mu(I_w) \leq \mu(I_w)$.
- When w survives, $\mathsf{M}_\mu(I_w) = \mu(I_w)$.
- M_μ combines dynamics and randomness $\longrightarrow$ "phase transitions".

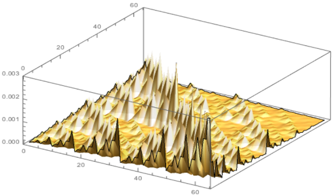
Questions: - Multifractal analysis of M_μ?
- Did we lose something (and what) on μ?

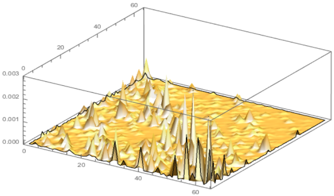
(Credits: F. Vigneron - UPEC)





2D Multinomial measure at generation 7 before and after sampling ($\eta = 0.8$)





2D Gibbs measure at generation 6 before and after sampling ($\eta = 0.7$)

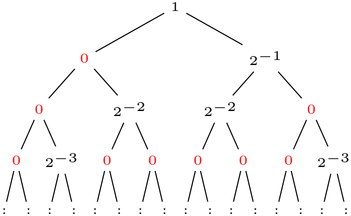
Outline of the rest of the talk:

- The Lebesgue case, with a proof !
- Recalls on Gibbs measures.
- Reconstruction of the tree.
- Multifractal analysis of M_μ.
- Main ideas of the proof.

Connections with :

- Dynamics
- Random covering questions.
- Diophantine approximation.

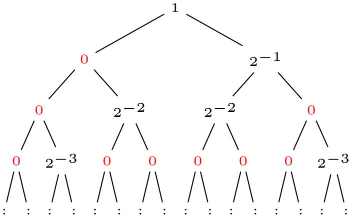
2 - The Lebesgue case

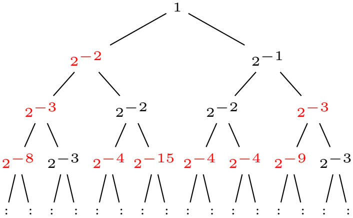


With $\mu = \lambda =$ Lebesgue, each vertex at generation j has a weight 2^{-j}.

The same for all survivors!

2 - The Lebesgue case

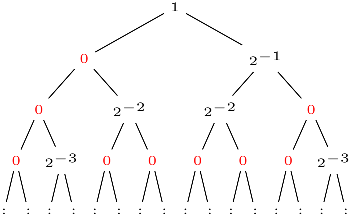


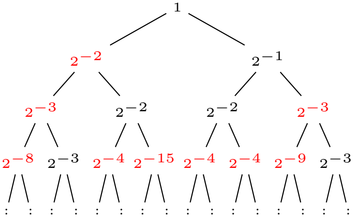


With $\mu = \lambda =$ Lebesgue, each vertex at generation j has a weight 2^{-j}.

The same for all survivors!

2 - The Lebesgue case





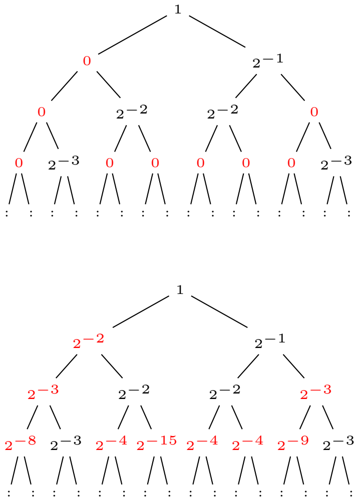
With $\mu = \lambda =$ Lebesgue,
each vertex at generation j
has a weight 2^{-j}.

The same for all survivors!

The value of $\mathsf{M}_\lambda(I_W)$
at a vertex W of length J
depends only on the generation
of the first survivor
amongst the sons of W.

But $\mathsf{M}_\lambda(I_W)$ does not depend on the "horizontal" location of this first survivor.

2 - The Lebesgue case



With $\mu = \lambda =$ Lebesgue,
each vertex at generation j
has a weight 2^{-j}.

The same for all survivors!

The value of $\mathsf{M}_\lambda(I_W)$
at a vertex W of length J
depends only on the generation
of the first survivor
amongst the sons of W.

But $\mathsf{M}_\lambda(I_W)$ does not depend on the "horizontal" location of this first survivor.

This model was studied by S. Jaffard
as lacunary wavelet series,
(also $\sim$ Lévy processes).

Question: can one compute the local dimension of M_μ at every x

$$\underline{\dim}(\mathsf{M}_\mu, x) = \liminf_{j\to+\infty} \frac{\log_2 \mathsf{M}_\mu(I_j(x))}{-j}$$

and the multifractal spectrum of M_μ defined by

$$D_{\mathsf{M}_\mu}(H) = \dim \left\{x \in [0,1] : \underline{\dim}(\mathsf{M}_\mu, x) = H\right\} \quad ?$$

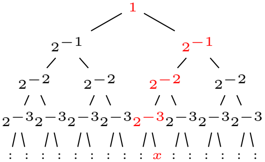
Question: can one compute the local dimension of M_μ at every x

$$\underline{\dim}(\mathsf{M}_\mu, x) = \liminf_{j \to +\infty} \frac{\log_2 \mathsf{M}_\mu(I_j(x))}{-j}$$

and the multifractal spectrum of M_μ defined by

$$D_{\mathsf{M}_\mu}(H) = \dim \left\{ x \in [0,1] : \underline{\dim}(\mathsf{M}_\mu, x) = H \right\} \quad ?$$

- For Lebesgue $\mu = \lambda$:



For all $x \in [0,1]$,
$\underline{\dim}(\lambda, x) = 1$.

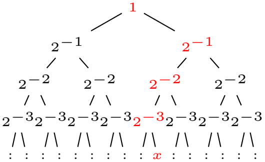
Question: can one compute the local dimension of M_μ at every x

$$\underline{\dim}(\mathsf{M}_\mu, x) = \liminf_{j\to+\infty} \frac{\log_2 \mathsf{M}_\mu(I_j(x))}{-j}$$

and the multifractal spectrum of M_μ defined by

$$D_{\mathsf{M}_\mu}(H) = \dim\left\{x \in [0,1] : \underline{\dim}(\mathsf{M}_\mu, x) = H\right\} \quad ?$$

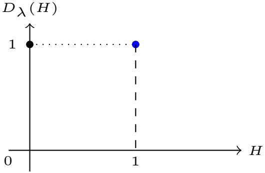
- For Lebesgue $\mu = \lambda$:



For all $x \in [0,1]$, $\underline{\dim}(\lambda, x) = 1$.

So $D_\lambda(1) = 1$,

and this is it!



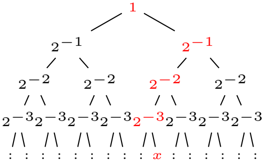
Question: can one compute the local dimension of M_μ at every x

$$\underline{\dim}(\mathsf{M}_\mu, x) = \liminf_{j\to+\infty} \frac{\log_2 \mathsf{M}_\mu(I_j(x))}{-j}$$

and the multifractal spectrum of M_μ defined by

$$D_{\mathsf{M}_\mu}(H) = \dim \left\{ x \in [0,1] : \underline{\dim}(\mathsf{M}_\mu, x) = H \right\} \quad ?$$

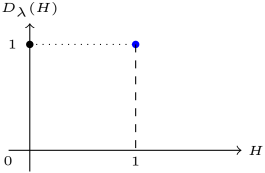
- For Lebesgue $\mu = \lambda$:



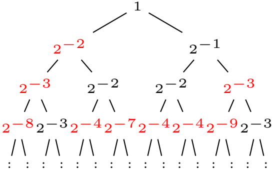
For all $x \in [0,1]$, $\underline{\dim}(\lambda, x) = 1$.

So $D_\lambda(1) = 1$,

and this is it!



- For the associated random capacity M_λ [Jaffard, 1999]:



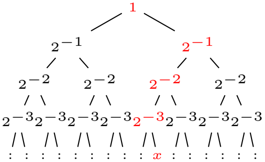
Question: can one compute the local dimension of M_μ at every x

$$\underline{\dim}(\mathsf{M}_\mu, x) = \liminf_{j\to+\infty} \frac{\log_2 \mathsf{M}_\mu(I_j(x))}{-j}$$

and the multifractal spectrum of M_μ defined by

$$D_{\mathsf{M}_\mu}(H) = \dim \left\{ x \in [0,1] : \underline{\dim}(\mathsf{M}_\mu, x) = H \right\} \quad ?$$

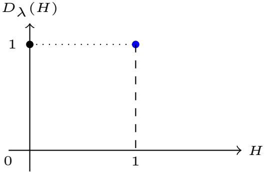
- For Lebesgue $\mu = \lambda$:



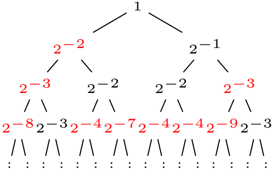
For all $x \in [0,1]$, $\underline{\dim}(\lambda, x) = 1$.

So $D_\lambda(1) = 1$,

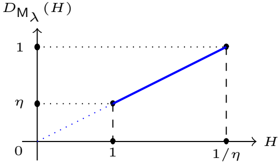
and this is it!



- For the associated random capacity M_λ [Jaffard, 1999]:



One has $D_{\mathsf{M}_\lambda}(H) = \eta H$ for every $H \in [1, 1/\eta]$, almost surely.



Main ideas of the proof in the Lebesgue case

We investigate the distribution of the surviving vertices forgetting all the ε's!

Each vertex w amongst the 2^j vertices at generation j is kept with proba $2^{-j(1-\eta)}$.

Main ideas of the proof in the Lebesgue case

We investigate the distribution of the surviving vertices forgetting all the ε's!

Each vertex w amongst the 2^j vertices at generation j is kept with proba $2^{-j(1-\eta)}$.

Lemma

Define $\mathcal{S}_j(\eta) = \{w \in \Sigma_j : w \text{ survives after sampling}\}$.

Main ideas of the proof in the Lebesgue case

We investigate the distribution of the surviving vertices forgetting all the ε's!

Each vertex w amongst the 2^j vertices at generation j is kept with proba $2^{-j(1-\eta)}$.

Lemma

Define $\mathcal{S}_j(\eta) = \{w \in \Sigma_j : w \text{ survives after sampling}\}$.

With probability 1, for every j,

$$\#\mathcal{S}_j(\eta) \sim 2^{j\eta},$$

and such that for every $W \in \Sigma_{j\eta}$,

$$\#(\mathcal{S}_j(\eta) \cap I_W) \sim 1.$$

Main ideas of the proof in the Lebesgue case

We investigate the distribution of the surviving vertices forgetting all the ε's!

Each vertex w amongst the 2^j vertices at generation j is kept with proba $2^{-j(1-\eta)}$.

Lemma

Define $\mathcal{S}_j(\eta) = \{w \in \Sigma_j : w \text{ survives after sampling}\}$.

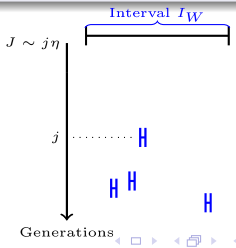
With probability 1, for every j,

$$\#\mathcal{S}_j(\eta) \sim 2^{j\eta},$$

and such that for every $W \in \Sigma_{j\eta}$,

$$\#(\mathcal{S}_j(\eta) \cap I_W) \sim 1.$$

In other words,
if $J = j\eta$,
every dyadic interval I_W
of generation J
contains at least one
(and essentially only one)
survivor at generation j.



Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

- First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

- First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

- For $j \geq 1$, $x \in I_{\lfloor j\eta \rfloor}(x)$, which contains a survivor w at generation j. Hence, one has

$$d(x, x_w) \leq 2^{-j\eta}.$$

Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

- First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

- For $j \geq 1$, $x \in I_{\lfloor j\eta \rfloor}(x)$, which contains a survivor w at generation j. Hence, one has

$$d(x, x_w) \leq 2^{-j\eta}.$$

So we have proved the covering property

almost surely, $$[0,1] = \limsup_{j\to+\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big)$$ (ε's are needed)

Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

• First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

• For $j \geq 1$, $x \in I_{\lfloor j\eta \rfloor}(x)$, which contains a survivor w at generation j. Hence, one has

$$d(x, x_w) \leq 2^{-j\eta}.$$

So we have proved the covering property

$$\text{almost surely,} \quad [0,1] = \limsup_{j\to+\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big) \qquad (\varepsilon\text{'s are needed})$$

Consequences on the local dimension of M_λ at x:

Again, the interval $I_{\lfloor j\eta \rfloor}(x)$ contains a survivor w, hence

$$\mathsf{M}_\lambda\Big(I_{\lfloor j\eta \rfloor}(x)\Big) \geq \lambda(I_w) = 2^{-j},$$

Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

• First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

• For $j \geq 1$, $x \in I_{\lfloor j\eta \rfloor}(x)$, which contains a survivor w at generation j. Hence, one has

$$d(x, x_w) \leq 2^{-j\eta}.$$

So we have proved the covering property

$$\text{almost surely,} \quad [0,1] = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big) \qquad (\varepsilon\text{'s are needed})$$

Consequences on the local dimension of M_λ at x:

Again, the interval $I_{\lfloor j\eta \rfloor}(x)$ contains a survivor w, hence

$$\mathsf{M}_\lambda\Big(I_{\lfloor j\eta \rfloor}(x)\Big) \geq \lambda(I_w) = 2^{-j},$$

which implies that

$$\underline{\dim}(\mathsf{M}_\lambda, x) = \liminf_{j\to\infty} \frac{\log_2 \mathsf{M}_\lambda\Big(I_j(x)\Big)}{-j} \leq \liminf_{j\to\infty} \frac{\log_2 2^{-j}}{-\lfloor j\eta \rfloor} = 1/\eta.$$

Changing viewpoint: Focus on $x \in [0,1]$, and the local behavior of M_λ at x.

• First, since $\mathsf{M}_\lambda(I) \leq \lambda(I)$, one obviously has $\underline{\dim}(\mathsf{M}_\lambda, x) \geq \underline{\dim}(\lambda, x) = 1$.

• For $j \geq 1$, $x \in I_{\lfloor j\eta \rfloor}(x)$, which contains a survivor w at generation j. Hence, one has

$$d(x, x_w) \leq 2^{-j\eta}.$$

So we have proved the covering property

$$\text{almost surely,} \quad [0,1] = \limsup_{j \to +\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\left(x_w, 2^{-j\eta}\right) \qquad (\varepsilon\text{'s are needed})$$

Consequences on the local dimension of M_λ at x:

Again, the interval $I_{\lfloor j\eta \rfloor}(x)$ contains a survivor w, hence

$$\mathsf{M}_\lambda\left(I_{\lfloor j\eta \rfloor}(x)\right) \geq \lambda(I_w) = 2^{-j},$$

which implies that

$$\underline{\dim}(\mathsf{M}_\lambda, x) = \liminf_{j \to \infty} \frac{\log_2 \mathsf{M}_\lambda\left(I_j(x)\right)}{-j} \leq \liminf_{j \to \infty} \frac{\log_2 2^{-j}}{-\lfloor j\eta \rfloor} = 1/\eta.$$

So we just proved:

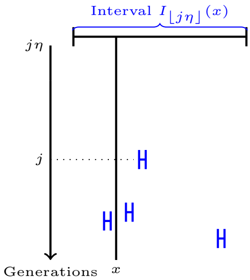
Lemma

For every $x \in [0,1]$, $\underline{\dim}(\mathsf{M}_\lambda, x) \in [1, 1/\eta]$.

For free, one has even more.

For free, one has even more.

We have seen that $\mathsf{M}_\lambda\left(I_{\lfloor j\eta \rfloor}(x)\right) \geq 2^{-j}$.

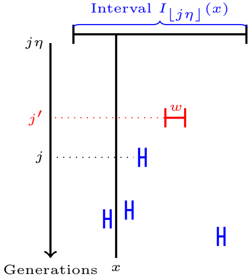


For free, one has even more.

We have seen that $\mathsf{M}_\lambda\left(I_{\lfloor j\eta\rfloor}(x)\right) \geq 2^{-j}$.

There may exist survivors

at a generation smaller than j.



For free, one has even more.

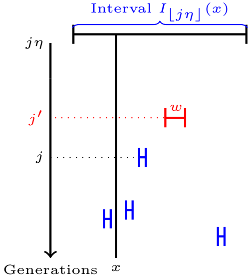
We have seen that $\mathsf{M}_\lambda\Big(I_{\lfloor j\eta\rfloor}(x)\Big) \geq 2^{-j}$.

There may exist survivors
at a generation smaller than j.

Let $w \in \mathcal{S}_{j'}(\eta)$, where $j' \in [\lfloor j\eta\rfloor, j]$, be
the first survivor in $I_{\lfloor j\eta\rfloor}(x)$.

Write $j = j'\,\eta\,\delta_j$, with $\delta_j \in [1, 1/\eta]$.

Obviously $\mathsf{M}_\lambda\Big(I_{\lfloor j\eta\rfloor}(x)\Big) = 2^{-j'} = 2^{-j/(\eta\delta_j)}$.



The same argument as before gives $\underline{\dim}(\mathsf{M}_\lambda, x) \leq \liminf_{j\to+\infty} \dfrac{1}{\eta\delta_j}$

For free, one has even more.

We have seen that $\mathsf{M}_\lambda\left(I_{\lfloor j\eta\rfloor}(x)\right) \geq 2^{-j}$.

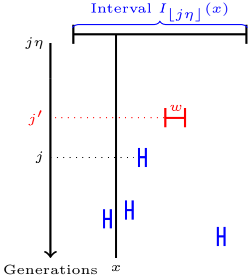
There may exist survivors

at a generation smaller than j.

Let $w \in \mathcal{S}_{j'}(\eta)$, where $j' \in [\lfloor j\eta\rfloor, j]$, be

the first survivor in $I_{\lfloor j\eta\rfloor}(x)$.

Write $j = j'\,\eta\,\delta_j$, with $\delta_j \in [1, 1/\eta]$.



Obviously $\mathsf{M}_\lambda\left(I_{\lfloor j\eta\rfloor}(x)\right) = 2^{-j'} = 2^{-j/(\eta\delta_j)}$.

The same argument as before gives $\underline{\dim}(\mathsf{M}_\lambda, x) \leq \liminf_{j\to+\infty} \dfrac{1}{\eta\delta_j}$.

Definition

Approximation rate of x by the random survivors: $\delta_x = \limsup_{j\to\infty} \delta_j$

For free, one has even more.

We have seen that $\mathsf{M}_\lambda\left(I_{\lfloor j\eta\rfloor}(x)\right) \geq 2^{-j}$.

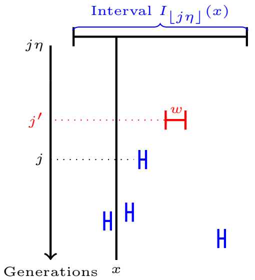
There may exist survivors

at a generation smaller than j.

Let $w \in \mathcal{S}_{j'}(\eta)$, where $j' \in [\lfloor j\eta\rfloor, j]$, be

the first survivor in $I_{\lfloor j\eta\rfloor}(x)$.

Write $j = j'\,\eta\,\delta_j$, with $\delta_j \in [1, 1/\eta]$.



Obviously $\mathsf{M}_\lambda\left(I_{\lfloor j\eta\rfloor}(x)\right) = 2^{-j'} = 2^{-j/(\eta\delta_j)}$.

The same argument as before gives $\underline{\dim}(\mathsf{M}_\lambda, x) \leq \liminf_{j\to+\infty} \dfrac{1}{\eta\delta_j}$.

Definition

Approximation rate of x by the random survivors: $\delta_x = \limsup_{j\to\infty} \delta_j$

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \dfrac{1}{\eta}\right]$ *and* $\underline{\dim}(\mathsf{M}_\lambda, x) = \dfrac{1}{\eta \cdot \delta_x}$.

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ and $\underline{\dim}(\mathsf{M}_\lambda, x) = \frac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big).$$

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ and $\underline{\dim}(\mathsf{M}_\lambda, x) = \frac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j \to +\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\left(x_w, 2^{-j\eta}\right).$$

Theorem (Dodson *et al.*, Jaffard, Beresnevich-Velani)

If for a (deterministic) sequence of balls $\left(B(x_n, l_n)\right)_n$ in $[0,1]$ one has

$$Leb\left(\limsup_{n \to +\infty} B(x_n, l_n)\right) = 1,$$

then for every $\delta \geq 1$, $\dim\left\{x \in [0,1] : \delta_x = \delta\right\} \geq \frac{1}{\delta}$ (here there is equality).

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ *and* $\underline{\dim}(\mathsf{M}_\lambda, x) = \frac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big).$$

Theorem (Dodson *et al.*, Jaffard, Beresnevich-Velani)

If for a (deterministic) sequence of balls $\Big(B(x_n, l_n)\Big)_n$ *in* $[0,1]$ *one has*

$$Leb\Big(\limsup_{n\to+\infty} B(x_n, l_n)\Big) = 1,$$

then for every $\delta \geq 1$, $\dim \left\{x \in [0,1] : \delta_x = \delta\right\} \geq \frac{1}{\delta}$ *(here there is equality).*

One concludes: Let $H \in [1, 1/\eta]$. Then

$D_{\mathsf{M}_\lambda}(H) = \dim \left\{x : \underline{\dim}(\mathsf{M}_\lambda, x) = H\right\}$

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ and $\underline{\dim}(\mathsf{M}_\lambda, x) = \frac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B\Big(x_w, 2^{-j\eta}\Big).$$

Theorem (Dodson *et al.*, Jaffard, Beresnevich-Velani)

If for a (deterministic) sequence of balls $\Big(B(x_n, l_n)\Big)_n$ in $[0,1]$ one has

$$Leb\Big(\limsup_{n\to+\infty} B(x_n, l_n)\Big) = 1,$$

then for every $\delta \geq 1$, $\dim\left\{x \in [0,1] : \delta_x = \delta\right\} \geq \frac{1}{\delta}$ (here there is equality).

One concludes: Let $H \in [1, 1/\eta]$. Then

$$D_{\mathsf{M}_\lambda}(H) = \dim\left\{x : \underline{\dim}(\mathsf{M}_\lambda, x) = H\right\} = \dim\left\{x : \frac{1}{\eta\delta_x} = H\right\}$$

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ and $\underline{\dim}(\mathsf{M}_\lambda, x) = \frac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j \to +\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\left(x_w, 2^{-j\eta}\right).$$

Theorem (Dodson *et al.*, Jaffard, Beresnevich-Velani)

If for a (deterministic) sequence of balls $\left(B(x_n, l_n)\right)_n$ in $[0,1]$ one has

$$Leb\left(\limsup_{n \to +\infty} B(x_n, l_n)\right) = 1,$$

then for every $\delta \geq 1$, $\dim \left\{x \in [0,1] : \delta_x = \delta\right\} \geq \frac{1}{\delta}$ (here there is equality).

One concludes: Let $H \in [1, 1/\eta]$. Then

$$D_{\mathsf{M}_\lambda}(H) = \dim \left\{x : \underline{\dim}(\mathsf{M}_\lambda, x) = H\right\} = \dim \left\{x : \frac{1}{\eta \delta_x} = H\right\}$$

$$= \dim \left\{x : \delta_x = (\eta H)^{-1}\right\}$$

Proposition

With probability 1, for every x, $\delta_x \in \left[1, \frac{1}{\eta}\right]$ and $\underline{\dim}(\mathsf{M}_\lambda, x) = \dfrac{1}{\eta \cdot \delta_x}$.

Recall that one has

$$[0,1] = \limsup_{j\to+\infty} \bigcup_{w \in \mathcal{S}_j(\eta)} B\left(x_w, 2^{-j\eta}\right).$$

Theorem (Dodson *et al.*, Jaffard, Beresnevich-Velani)

If for a (deterministic) sequence of balls $\Big(B(x_n, l_n)\Big)_n$ in $[0,1]$ one has

$$Leb\left(\limsup_{n\to+\infty} B(x_n, l_n)\right) = 1,$$

then for every $\delta \geq 1$, $\dim\left\{x \in [0,1] : \delta_x = \delta\right\} \geq \dfrac{1}{\delta}$ (here there is equality).

One concludes: Let $H \in [1, 1/\eta]$. Then

$$D_{\mathsf{M}_\lambda}(H) = \dim\left\{x : \underline{\dim}(\mathsf{M}_\lambda, x) = H\right\} = \dim\left\{x : \frac{1}{\eta\delta_x} = H\right\}$$

$$= \dim\left\{x : \delta_x = (\eta H)^{-1}\right\} = \frac{1}{(\eta H)^{-1}} = \eta H.$$

3 - Gibbs measures, and reconstruction of the initial tree

- Consider a Hölder potential $\varphi : \Sigma \to \mathbb{R}$.
- Consider the shift σ on Σ: $\sigma(w_1 w_2 w_3 ...) = w_2 w_3 ....$
- Look at the Birkhoff sums $S_j\varphi(t) = \sum_{k=0}^{j-1} \varphi(\sigma^k t)$.
- The pressure is $P(\varphi) = \lim_{j\to\infty} \frac{1}{j} \log \sum_{w\in\Sigma_j} \sup_{t\in[w]} e^{S_j\varphi(t)} \in \mathbb{R}$.

3 - Gibbs measures, and reconstruction of the initial tree

- Consider a Hölder potential $\varphi : \Sigma \to \mathbb{R}$.
- Consider the shift σ on Σ: $\sigma(w_1 w_2 w_3 ...) = w_2 w_3 ...$.
- Look at the Birkhoff sums $S_j\varphi(t) = \sum_{k=0}^{j-1} \varphi(\sigma^k t)$.
- The pressure is $P(\varphi) = \lim_{j\to\infty} \frac{1}{j} \log \sum_{w\in\Sigma_j} \sup_{t\in[w]} e^{S_j\varphi(t)} \in \mathbb{R}$.

Theorem

There is a Gibbs measure μ_φ defined on Σ, such that

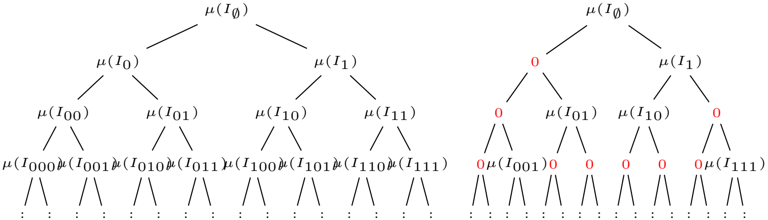
$$\exists C > 1 : \quad \forall x \in [0,1], \quad \forall\, j \geq 1, \quad C^{-1} \leq \frac{\mu_\varphi(I_j(x))}{\exp\left(S_j\varphi(x) - jP(\varphi)\right)} \leq C,$$

This measure satisfies the quasi-Bernoulli property:

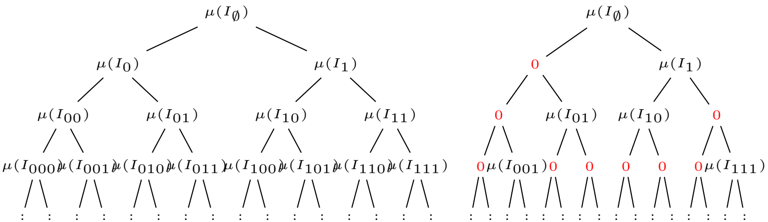
$$\forall\,(w,w') \in \Sigma_j \times \Sigma_{j'}, \quad C^{-1}\mu(I_w)\mu(I_{w'}) \leq \mu(I_{w\cdot w'}) \leq C\mu(I_w)\mu(I_{w'}).$$

The multiplicativity is key in the following !!

Fix a Gibbs measure μ, a sampling index $0 < \eta < 1$. Reconstruction of μ?



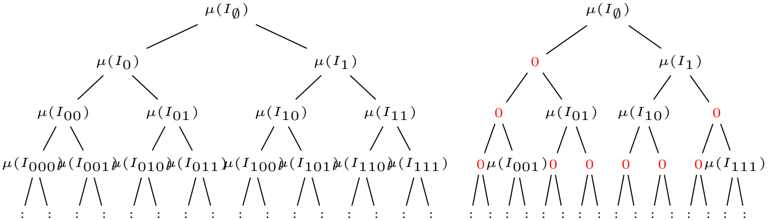
Fix a Gibbs measure μ, a sampling index $0 < \eta < 1$. Reconstruction of μ?



Definition

A finite word u is 1-reconstructible if there exists a (finite) word w such that both w and wu survive.

Fix a Gibbs measure μ, a sampling index $0 < \eta < 1$. Reconstruction of μ?

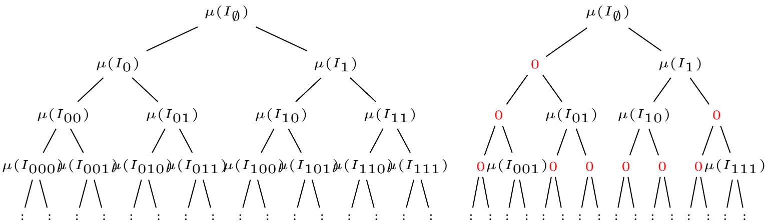


Definition

A finite word u is 1-reconstructible if there exists a (finite) word w such that both w and wu survive.

Using the quasi-Bernoulli property $\mu(I_{wu}) \sim \mu(I_w)\mu(I_u)$. When u is 1-reconstructible, one deduces the value $\mu(I_u)$ up to a constant.

Fix a Gibbs measure μ, a sampling index $0 < \eta < 1$. Reconstruction of μ?



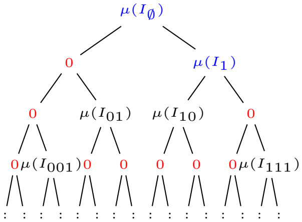
Definition

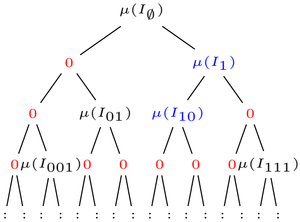
A finite word u is 1-reconstructible if there exists a (finite) word w such that both w and wu survive.

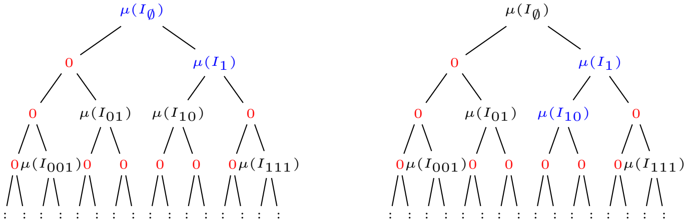
Using the quasi-Bernoulli property $\mu(I_{wu}) \sim \mu(I_w)\mu(I_u)$. When u is 1-reconstructible, one deduces the value $\mu(I_u)$ up to a constant.

Definition

A finite word u is k-reconstructible if there exists a decomposition $u = u_1 u_2 \cdots u_k$ such that all $u_1 ..., u_k$ are simultaneously 1-reconstructible.


$\mu(I_\emptyset)$
0
$\mu(I_1)$
0
$\mu(I_{01})$
$\mu(I_{10})$
0
0 $\mu(I_{001})$ 0 0 0 0 0 $\mu(I_{111})$

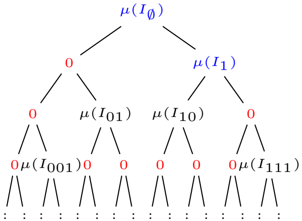

$\mu(I_\emptyset)$
0
$\mu(I_1)$
0
$\mu(I_{01})$
$\mu(I_{10})$
0
0 $\mu(I_{001})$ 0 0 0 0 0 $\mu(I_{111})$

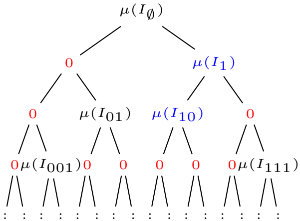


Theorem

If $\eta > 1/2$, then a.s. the initial Gibbs tree is 1-reconstructible;

If $\eta < 1/2$, then a.s. the initial Gibbs tree not k-reconstructible, for any $k \geq 1$.





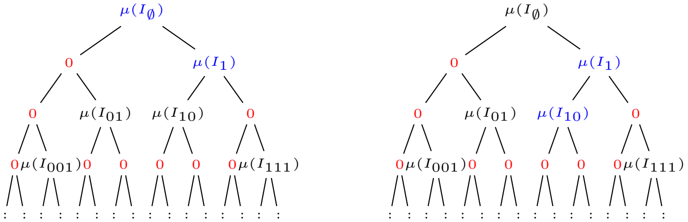
Theorem

If $\eta > 1/2$, then a.s. the initial Gibbs tree is 1-reconstructible;

If $\eta < 1/2$, then a.s. the initial Gibbs tree not k-reconstructible, for any $k \geq 1$.

It follows from the fact that, if $|w| = j$ and $|u| = k$, then

$$\mathbb{P}\Big(\text{both } w \text{ and } wu \text{ survive}\Big) = 2^{-\eta k} 2^{-2(1-\eta)j}.$$



Theorem

If $\eta > 1/2$, then a.s. the initial Gibbs tree is 1-reconstructible;

If $\eta < 1/2$, then a.s. the initial Gibbs tree not k-reconstructible, for any $k \geq 1$.

It follows from the fact that, if $|w| = j$ and $|u| = k$, then

$$\mathbb{P}\Big(\text{both } w \text{ and } wu \text{ survive}\Big) = 2^{-\eta k} 2^{-2(1-\eta)j}.$$

Phase transition at $\eta = 1/2$.

4- Multifractal analysis of the random capacity M_μ

Recall that the L^q-scaling function (free energy) of μ is :

$$\text{for } q \in \mathbb{R} \ , \quad \tau_\mu(q) = \lim_{j\to\infty} \frac{-1}{j} \log_2 \sum_{w\in\Sigma_j} \mu(I_w)^q .$$

τ_μ is real analytic.

4- Multifractal analysis of the random capacity M_μ

Recall that the L^q-scaling function (free energy) of μ is :

$$\text{for } q \in \mathbb{R}\ , \quad \tau_\mu(q) = \lim_{j\to\infty} \frac{-1}{j} \log_2 \sum_{w\in\Sigma_j} \mu(I_w)^q.$$

τ_μ is real analytic. The Legendre transform of τ_μ is defined by

$$(\tau_\mu)^*(H) \ = \ \inf\{Hq - \tau_\mu(q) : q \in \mathbb{R}\}.$$

4- Multifractal analysis of the random capacity M_μ

Recall that the L^q-scaling function (free energy) of μ is :

$$\text{for } q \in \mathbb{R}\ , \quad \tau_\mu(q) = \lim_{j\to\infty} \frac{-1}{j} \log_2 \sum_{w\in\Sigma_j} \mu(I_w)^q.$$

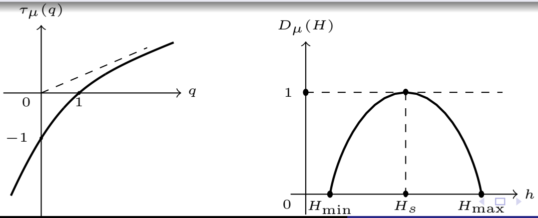
τ_μ is real analytic. The Legendre transform of τ_μ is defined by

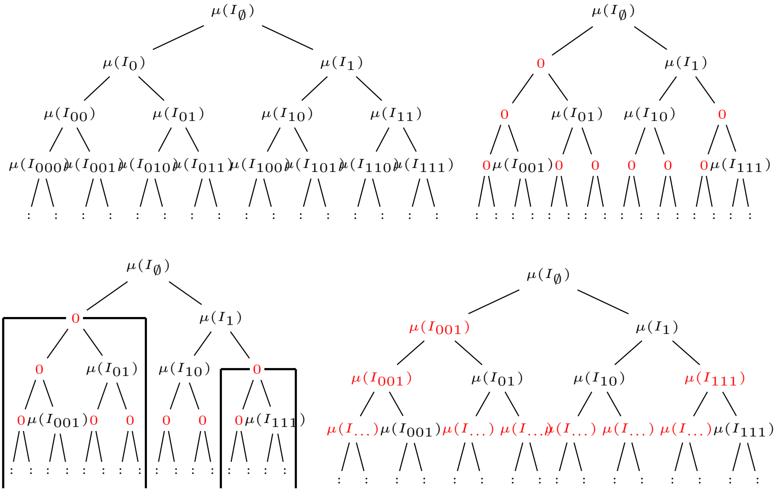
$$(\tau_\mu)^*(H) \ = \ \inf\{Hq - \tau_\mu(q) : q \in \mathbb{R}\}.$$

Theorem (Collet-Lebowitz-Porzio, '87)

Let $H_{\min} = \tau_\mu'(+\infty)$, $H_s = \tau_\mu'(0)$ and $H_{\max} = \tau_\mu'(-\infty)$.

- *for μ-almost every x, $\underline{\dim}(\mu, x) = \dim \mu = \tau_\mu'(1)$.*
- *For every $H \in [H_{\min}, H_{\max}]$, $\quad D_\mu(H) = (\tau_\mu)^*(H) \geq 0$.*
- *If $H \notin [H_{\min}, H_{\max}]$, then $\{x : \underline{\dim}(\mu, x) = H\} = \emptyset$.*




μ(I_∅)
μ(I_0) μ(I_1)
μ(I_00) μ(I_01) μ(I_10) μ(I_11)
μ(I_000) μ(I_001) μ(I_010) μ(I_011) μ(I_100) μ(I_101) μ(I_110) μ(I_111)
μ(I_∅)
0 μ(I_1)
0 μ(I_01) μ(I_10) 0
0 μ(I_001) 0 0 0 0 0 μ(I_111)
μ(I_∅)
0 μ(I_1)
0 μ(I_01) μ(I_10) 0
0 μ(I_001) 0 0 0 0 0 μ(I_111)
μ(I_∅)
μ(I_001) μ(I_1)
μ(I_001) μ(I_01) μ(I_10) μ(I_111)
μ(I_...) μ(I_001) μ(I_...) μ(I_...) μ(I_...) μ(I_...) μ(I_...) μ(I_111)

Theorem (Barral, S., 2015)

There exist $\widetilde{\eta} \in (0,\eta)$ *and* $H_\ell(\widetilde{\eta}) \in [H_{\min}, H_s]$ *such that, with probability 1:*

1. *The free energy* τ_{M_μ} *of* M_μ *exists as a limit.*
2. *The spectrum of singularity of* M_μ *is:*

$$D_{\mathsf{M}_\mu}(H) = \begin{cases} D_\mu(H) - (1-\eta) & \text{if} \quad H_\ell(0) \leq H \leq H_\ell(\widetilde{\eta}), \\ \dfrac{\widetilde{\eta}}{H_\ell(\widetilde{\eta})} D_\mu(H_\ell(\widetilde{\eta})) \cdot H & \text{if} \quad H_\ell(\widetilde{\eta}) \leq H \leq H_\ell(\widetilde{\eta})/\widetilde{\eta}, \\ D_\mu(H - \frac{1-\widetilde{\eta}}{\widetilde{\eta}} H_\ell(\widetilde{\eta})) & \text{if} \quad H_\ell(\widetilde{\eta})/\widetilde{\eta} \leq H \leq H_{\max} + \frac{1-\widetilde{\eta}}{\widetilde{\eta}} H_\ell(\widetilde{\eta}). \end{cases}$$

3. M_μ *verifies the multifractal formalism:* $D_{\mathsf{M}_\mu} = (\tau_{\mathsf{M}_\mu})^*$.

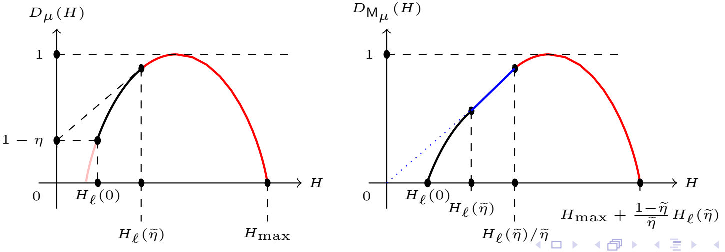
Theorem (Barral, S., 2015)

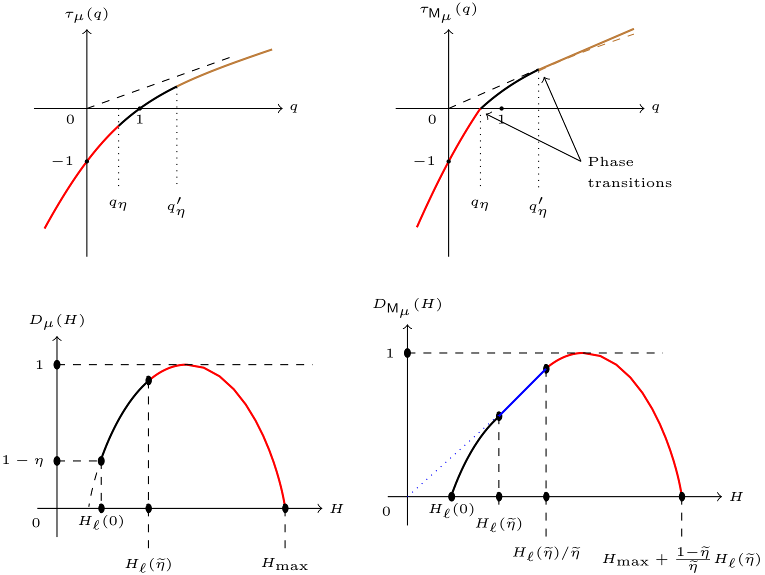
There exist $\widetilde{\eta} \in (0, \eta)$ *and* $H_\ell(\widetilde{\eta}) \in [H_{\min}, H_s]$ *such that, with probability 1:*

1. *The free energy* τ_{M_μ} *of* M_μ *exists as a limit.*
2. *The spectrum of singularity of* M_μ *is:*

$$D_{\mathsf{M}_\mu}(H) = \begin{cases} D_\mu(H) - (1-\eta) & \text{if} \quad H_\ell(0) \le H \le H_\ell(\widetilde{\eta}), \\ \dfrac{\widetilde{\eta}}{H_\ell(\widetilde{\eta})} D_\mu(H_\ell(\widetilde{\eta})) \cdot H & \text{if} \quad H_\ell(\widetilde{\eta}) \le H \le H_\ell(\widetilde{\eta})/\widetilde{\eta}, \\ D_\mu(H - \frac{1-\widetilde{\eta}}{\widetilde{\eta}} H_\ell(\widetilde{\eta})) & \text{if} \quad H_\ell(\widetilde{\eta})/\widetilde{\eta} \le H \le H_{\max} + \frac{1-\widetilde{\eta}}{\widetilde{\eta}} H_\ell(\widetilde{\eta}). \end{cases}$$

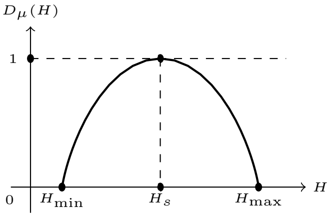
3. M_μ *verifies the multifractal formalism:* $D_{\mathsf{M}_\mu} = (\tau_{\mathsf{M}_\mu})^*$.




$\tau_\mu(q)$
q
0
1
-1
q_η
q'_η
$\tau_{M_\mu}(q)$
q
0
1
-1
q_η
q'_η
Phase
transitions
$D_\mu(H)$
1
$1-\eta$
0
H
$H_\ell(0)$
$H_\ell(\widetilde{\eta})$
$H_{\max}$
$D_{M_\mu}(H)$
1
0
H
$H_\ell(0)$
$H_\ell(\widetilde{\eta})$
$H_\ell(\widetilde{\eta})/\widetilde{\eta}$
$H_{\max} + \frac{1-\widetilde{\eta}}{\widetilde{\eta}} H_\ell(\widetilde{\eta})$

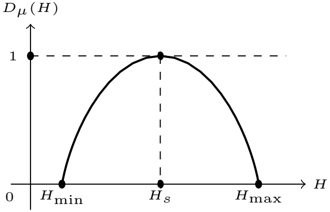
5 - Idea of the proof

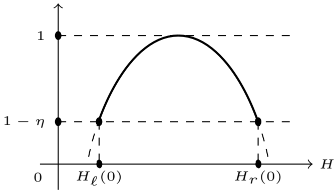
New parameters need to be introduced:



5 - Idea of the proof

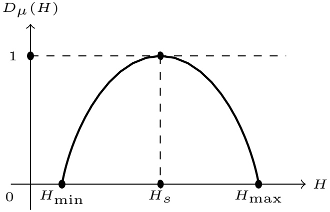
New parameters need to be introduced:

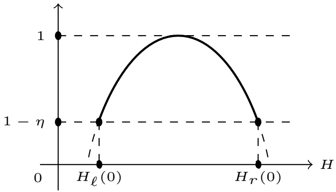




5 - Idea of the proof

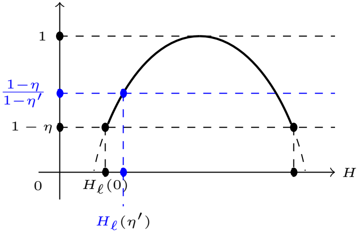
New parameters need to be introduced:





For every $\eta' \in [0, \eta]$,
one considers $H_\ell(\eta')$
the unique solution to

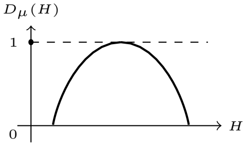
$$\mathbf{D}_\mu\Big(\mathbf{H}_\ell(\eta')\Big) = \frac{\mathbf{1} - \eta}{\mathbf{1} - \eta'}.$$



For a given H, set

$\mathcal{E}_\mu(j,H) = \{w \in \Sigma_j : \mu(I_w) \sim 2^{-jH}\}$.

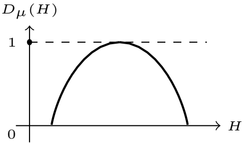
One has $\#\mathcal{E}_\mu(j,H) \sim 2^{jD_\mu(H)}$.



For a given H, set

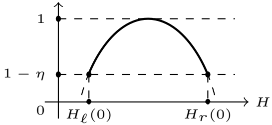
$\mathcal{E}_\mu(j,H) = \{w \in \Sigma_j : \mu(I_w) \sim 2^{-jH}\}$.

One has $\#\mathcal{E}_\mu(j,H) \sim 2^{jD_\mu(H)}$.



At a given generation j,

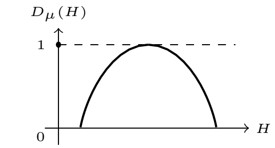
one keeps only $\sim 2^{j\eta}$ coefficients amongst the 2^j.



For a given H, set

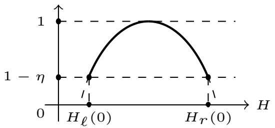
$\mathcal{E}_\mu(j,H) = \{w \in \Sigma_j : \mu(I_w) \sim 2^{-jH}\}$.

One has $\#\mathcal{E}_\mu(j,H) \sim 2^{jD_\mu(H)}$.



At a given generation j,

one keeps only $\sim 2^{j\eta}$ coefficients amongst the 2^j.



By a counting argument, one gets

Lemma

With probability 1:

- *Only those words w such that*

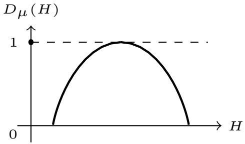
$2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$

may survive.

For a given H, set

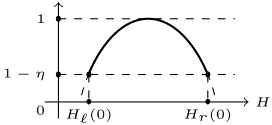
$\mathcal{E}_\mu(j,H) = \{w \in \Sigma_j : \mu(I_w) \sim 2^{-jH}\}$.

One has $\#\mathcal{E}_\mu(j,H) \sim 2^{jD_\mu(H)}$.



At a given generation j,

one keeps only $\sim 2^{j\eta}$ coefficients amongst the 2^j.

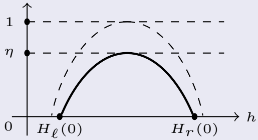


By a counting argument, one gets

Lemma

With probability 1:

- *Only those words w such that* $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ *may survive.*
- *For $H \in [H_\ell(0), H_r(0)]$, one has* $\#\mathcal{E}_\mu(j,H) \cap \mathcal{S}_j(\eta) \sim 2^{j\left(D_\mu(H)-(1-\eta)\right)}$.



Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Only the w such $2^{-jH_r(0)} \le \mu(I_w) \le 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \ge H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Proof: For every x and $J \geq 1$, in the interval $I_J(x)$, there exists a survivor $w \in \mathcal{S}_j(\eta)$ with $j \geq J$ such that $I_w \subset I_J(x)$ and $\mu(I_w)$ realizes the maximum. Jence

$$\mathsf{M}_\mu\Big(I_J(x)\Big) = \mu(I_w)$$

Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Proof: For every x and $J \geq 1$, in the interval $I_J(x)$, there exists a survivor $w \in \mathcal{S}_j(\eta)$ with $j \geq J$ such that $I_w \subset I_J(x)$ and $\mu(I_w)$ realizes the maximum. Jence

$$\mathsf{M}_\mu\Big(I_J(x)\Big) = \mu(I_w) \leq 2^{-jH_\ell(0)} \leq 2^{-JH_\ell(0)}.$$

Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Proof: For every x and $J \geq 1$, in the interval $I_J(x)$, there exists a survivor $w \in \mathcal{S}_j(\eta)$ with $j \geq J$ such that $I_w \subset I_J(x)$ and $\mu(I_w)$ realizes the maximum. Jence

$$\mathsf{M}_\mu\Big(I_J(x)\Big) = \mu(I_w) \leq 2^{-jH_\ell(0)} \leq 2^{-JH_\ell(0)}.$$

So for every J, $\dfrac{\log_2 \mathsf{M}_\mu(I_J(x))}{-J} \geq H_\ell(0)$.

and $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.

Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Proof: For every x and $J \geq 1$, in the interval $I_J(x)$, there exists a survivor $w \in \mathcal{S}_j(\eta)$ with $j \geq J$ such that $I_w \subset I_J(x)$ and $\mu(I_w)$ realizes the maximum. Jence

$$\mathsf{M}_\mu\Big(I_J(x)\Big) = \mu(I_w) \leq 2^{-jH_\ell(0)} \leq 2^{-JH_\ell(0)}.$$

So for every J, $\dfrac{\log_2 \mathsf{M}_\mu(I_J(x))}{-J} \geq H_\ell(0)$.

and $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.

For the existence of points x with local dimension $H_\ell(0)$, arguments of random coverings and "nice" distribution of random points are involved.

Only the w such $2^{-jH_r(0)} \leq \mu(I_w) \leq 2^{-jH_\ell(0)}$ may survive.

Immediate consequence:

Lemma

With probability 1, the capacity M_μ *satisfies:*

- *for every* x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.
- *there exists* x *such that* $\underline{\dim}(\mathsf{M}_\mu, x) = H_\ell(0)$.

Proof: For every x and $J \geq 1$, in the interval $I_J(x)$, there exists a survivor $w \in \mathcal{S}_j(\eta)$ with $j \geq J$ such that $I_w \subset I_J(x)$ and $\mu(I_w)$ realizes the maximum. Jence

$$\mathsf{M}_\mu\Big(I_J(x)\Big) = \mu(I_w) \leq 2^{-jH_\ell(0)} \leq 2^{-JH_\ell(0)}.$$

So for every J, $\dfrac{\log_2 \mathsf{M}_\mu(I_J(x))}{-J} \geq H_\ell(0)$.

and $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(0)$.

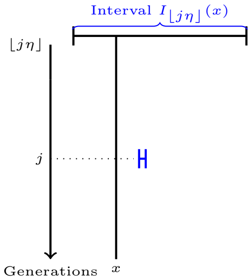
For the existence of points x with local dimension $H_\ell(0)$, arguments of random coverings and "nice" distribution of random points are involved.

Much more difficult to find an upper bound for $\underline{\dim}(\mathsf{M}_\mu, x)$.

This upper bound is $H_{\max} + \frac{1-\eta}{\eta} H_\ell(\tilde\eta) \gg H_{\max} \gg H_r(0)$!!

5 - Idea of the proof

We arrive at the core of the problem:



5 - Idea of the proof

We arrive at the core of the problem:

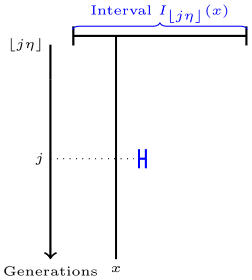
The "first" survivor

In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!



5 - Idea of the proof

We arrive at the core of the problem:

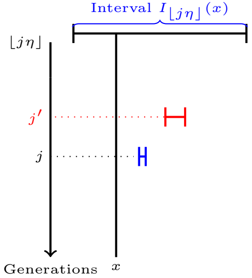
The "first" survivor

In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!



5 - Idea of the proof

We arrive at the core of the problem:

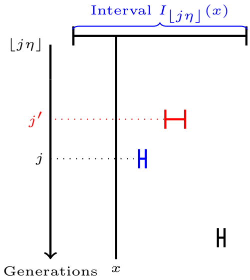
The "first" survivor

In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!



5 - Idea of the proof

We arrive at the core of the problem:

The "first" survivor

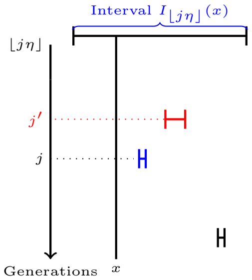
In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!

$\longrightarrow$ There is a **competition** between survivors: generation + local behavior of μ.



5 - Idea of the proof

We arrive at the core of the problem:

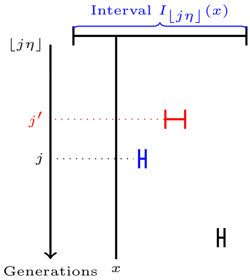
The "first" survivor

In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!



$\longrightarrow$ There is a **competition** between survivors: generation + local behavior of μ.

$\longrightarrow$ One needs to describe more precisely which local dimensions may survive, within an interval $I_{\lfloor j\eta \rfloor}(x)$.

Theorem

If for a (deterministic) sequence of balls $\Big(B(x_n, l_n)\Big)_n$ *in* $[0,1]$ *one has*

$$Leb\Big(\limsup_{n\to+\infty} B(x_n, l_n)\Big) = 1,$$

then for every $\delta \geq 1$, $\dim\left\{x \in [0,1] : \delta_x = \delta\right\} \geq \dfrac{1}{\delta}$.

5 - Idea of the proof

We arrive at the core of the problem:

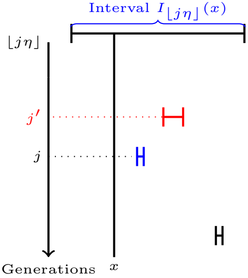
The "first" survivor

In the interval $I_{\lfloor j\eta \rfloor}(x)$

is probably not

the one that contributes most

to the value of $\mathsf{M}_\mu(I_{\lfloor j\eta \rfloor}(x))$!



$\longrightarrow$ There is a **competition** between survivors: generation + local behavior of μ.

$\longrightarrow$ One needs to describe more precisely which local dimensions may survive, within an interval $I_{\lfloor j\eta \rfloor}(x)$.

Theorem (Barral-S. 2004)

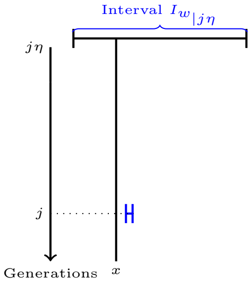
If for a (deterministic) sequence of balls $\Big(B(x_n, l_n)\Big)_n$ in $[0,1]$ one has

$$\mu\Big(\limsup_{n\to+\infty} B(x_n, l_n)\Big) = 1,$$

then for every $\delta \geq 1$, $\dim \left\{x \in [0,1] : \delta_x = \delta\right\} \geq \dfrac{\dim \mu}{\delta}$.

Focus on one survivor w of generation j.

Focus on one survivor w of generation j.



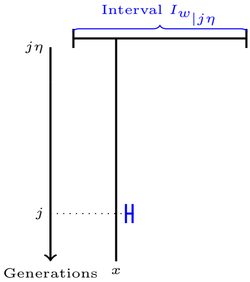
Focus on one survivor w of generation j.

Recall that σ is the shift, then

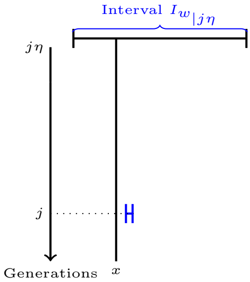
$\mu(I_w) \sim \mu(I_{w_{|j\eta}})\,\mu\big(I_{\sigma^{j\eta}w}\big)$,

where $w_{|j\eta}$ of length $j\eta$ is the η-root of w,

and $\sigma^{j\eta}w$ of length $j - j\eta$ is the η-tail of w.



Focus on one survivor w of generation j.



Recall that σ is the shift, then

$\mu(I_w) \sim \mu(I_{w_{|j\eta}})\,\mu\big(I_{\sigma^{j\eta}w}\big)$,

where $w_{|j\eta}$ of length $j\eta$ is the η-root of w,

and $\sigma^{j\eta}w$ of length $j - j\eta$ is the η-tail of w.

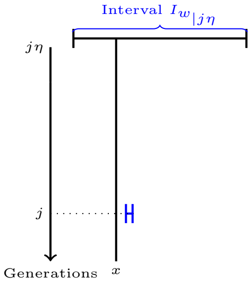
Hence

$\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

where

- α describes the scaling behavior of the η-root,
- β describes the scaling behavior of the η-tail.

Focus on one survivor w of generation j.



Recall that σ is the shift, then

$\mu(I_w) \sim \mu(I_{w_{|j\eta}})\, \mu\big(I_{\sigma^{j\eta} w}\big)$,

where $w_{|j\eta}$ of length $j\eta$ is the η-root of w,

and $\sigma^{j\eta} w$ of length $j - j\eta$ is the η-tail of w.

Hence

$\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

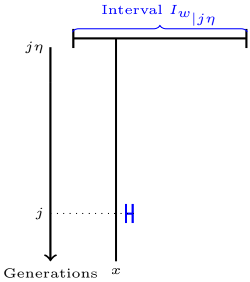
where

- α describes the scaling behavior of the η-root,
- β describes the scaling behavior of the η-tail.

This rewrites

$$H_w = \eta\alpha + (1-\eta)\beta.$$

Focus on one survivor w of generation j.



Recall that σ is the shift, then

$\mu(I_w) \sim \mu(I_{w_{|j\eta}})\,\mu(I_{\sigma^{j\eta}w})$,

where $w_{|j\eta}$ of length $j\eta$ is the η-root of w,

and $\sigma^{j\eta}w$ of length $j - j\eta$ is the η-tail of w.

Hence

$\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

where

- α describes the scaling behavior of the η-root,
- β describes the scaling behavior of the η-tail.

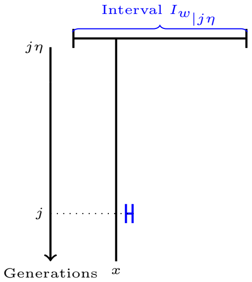
This rewrites

$$H_w = \eta\alpha + (1-\eta)\beta.$$

Each interval I_W, where W has length $J = j\eta$, contains a survivor at generation j.

Hence every $\alpha \in [H_{\min}, H_{\max}]$ is possible.

Focus on one survivor w of generation j.



Recall that σ is the shift, then

$\mu(I_w) \sim \mu(I_{w_{|j\eta}})\,\mu\big(I_{\sigma^{j\eta}w}\big),$

where $w_{|j\eta}$ of length $j\eta$ is the η-root of w,

and $\sigma^{j\eta}w$ of length $j - j\eta$ is the η-tail of w.

Hence

$\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta},$

where

- α describes the scaling behavior of the η-root,
- β describes the scaling behavior of the η-tail.

This rewrites

$$H_w = \eta\alpha + (1-\eta)\beta.$$

Each interval I_W, where W has length $J = j\eta$, contains a survivor at generation j.

Hence every $\alpha \in [H_{\min}, H_{\max}]$ is possible.

Question: Can we describe the possible β's?

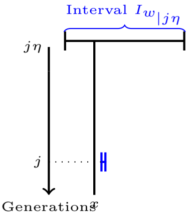
One has $\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

and so $H_w = \eta\alpha + (1-\eta)\beta$.

Since the location of w is random, one could think

that one exponent β is realized a.s.,

the same for all intervals $I_{w_{|j\eta}}$.



One has $\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

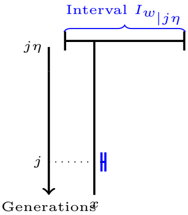
and so $H_w = \eta\alpha + (1-\eta)\beta$.

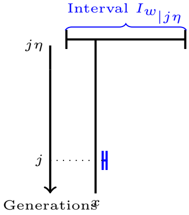
Since the location of w is random, one could think

that one exponent β is realized a.s.,

the same for all intervals $I_{w_{|j\eta}}$.

It is not true $\longrightarrow$ not easy to describe.





One has $\mu(I_w) = 2^{-jH_w} \sim 2^{-j\eta\alpha} \cdot 2^{-j(1-\eta)\beta}$,

and so $H_w = \eta\alpha + (1-\eta)\beta$.

Since the location of w is random, one could think

that one exponent β is realized a.s.,

the same for all intervals $I_{w_{|j\eta}}$.

It is not true $\longrightarrow$ not easy to describe.

One must consider all possible decompositions in tails and roots for $\eta' \in (0, \eta]$:

$$w \; = \; w_1\, w_2 \, \cdots \, w_{\lfloor j\eta' \rfloor} \;\; w_{\lfloor j\eta' \rfloor + 1}\, w_{\lfloor j\eta' \rfloor + 2} \, \cdots \, w_j$$

η'-root of w $\qquad$ η'-tail of w

$$\frac{\log_2 \mu\big(I_{w_{|\lfloor \eta' j \rfloor}}\big)}{-\lfloor \eta' j \rfloor} \sim \alpha \qquad\qquad \frac{\log_2 \mu\big(I_{\sigma^{\lfloor \eta' j \rfloor} w}\big)}{j - \lfloor \eta' j \rfloor} = ?$$

Lemma

With proba 1, for every survivor $w \in \Sigma_j$, there exists $\eta' \in [0,\eta]$ such that

$$\mu(I_w) \sim \mu(I_{w_{|j\eta'}}) \cdot 2^{-j(1-\eta')H_\ell(\eta')} \qquad \text{or the same with } H_r(\eta').$$

Lemma

With proba 1, for every survivor $w \in \Sigma_j$, there exists $\eta' \in [0,\eta]$ such that

$$\mu(I_w) \sim \mu(I_{w_{|j\eta'}}) \cdot 2^{-j(1-\eta')H_\ell(\eta')} \qquad \text{or the same with } H_r(\eta').$$

We have $\mu(I_w) = 2^{-jH_w}$, hence for some η',

$$H_w = \eta'\alpha + (1-\eta')H_\ell(\eta').$$

$$w = w_1 w_2 \cdots w_{\lfloor j\eta' \rfloor} \; w_{\lfloor j\eta' \rfloor+1} w_{\lfloor j\eta' \rfloor+2} \cdots w_j$$

η'-root of w $\qquad$ η'-tail of w

$$\frac{\log_2 \mu(I_{w_{|\lfloor \eta' j \rfloor}})}{-\lfloor \eta' j \rfloor} \sim \alpha \qquad\qquad \frac{\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})}{j-\lfloor \eta' j \rfloor} \sim H_\ell(\eta')$$

Lemma

With proba 1, for every survivor $w \in \Sigma_j$, there exists $\eta' \in [0,\eta]$ such that

$$\mu(I_w) \sim \mu(I_{w_{|j\eta'}}) \cdot 2^{-j(1-\eta')H_\ell(\eta')} \qquad \text{or the same with } H_r(\eta').$$

We have $\mu(I_w) = 2^{-jH_w}$, hence for some η',

$$H_w = \eta'\alpha + (1-\eta')H_\ell(\eta').$$

$$w = w_1 w_2 \cdots w_{\lfloor j\eta' \rfloor} \; w_{\lfloor j\eta' \rfloor + 1} w_{\lfloor j\eta' \rfloor + 2} \cdots w_j$$

η'-root of w (first part) — η'-tail of w (second part)

$$\frac{\log_2 \mu(I_{w_{|\lfloor \eta' j \rfloor}})}{-\lfloor \eta' j \rfloor} \sim \alpha \qquad\qquad \frac{\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})}{j - \lfloor \eta' j \rfloor} \sim H_\ell(\eta')$$

Lemma (Renewal property)

With proba 1, for every $\eta' \in [0,\eta]$ and every word W of generation $j\eta'$, there is a survivor w of generation j tel que

$$\mu(I_w) \sim \mu(I_W) \cdot 2^{-j(1-\eta')H_\ell(\eta')}.$$

Conclusion(s) :

• M_μ satisfies the multifractal formalism: for every H, $D_{\mathsf{M}_\mu}(H) = (\tau_{\mathsf{M}_\mu})^*(H)$.

• The phase transitons appear in the proof !

• Other energy models: cascades, random walks on trees.

• Other sampling procedures (less "radical") $\longrightarrow$ other phase transitions?

• General question: can one recover from partial information the initial "dynamics" or the original "measure".