Integrability of Deep Water Equations

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Basic equations

We study the potential flow of two-dimensional ideal incompressible fluid. The fluid occupies a half-infinite domain

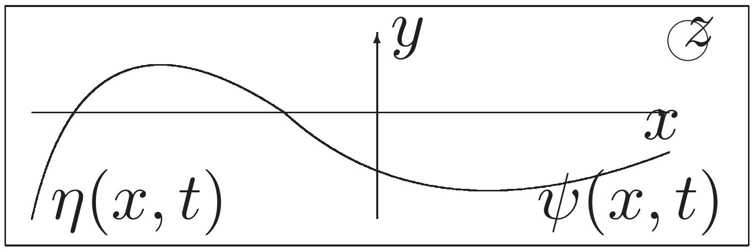
$$-\infty < y < \eta(x,t), \quad -\infty < x < \infty.$$

The flow is potential, so that $v = \nabla\Phi$, $\Phi|_{y=\eta(x,t)} = \psi(x,t)$. Boundary conditions on the surface are standard. It is known that the shape of surface $\eta(x,t)$ and the potential on the surface $\psi(x,t)$ form a pair of canonically conjugated variables obeying the Hamiltonian equations:

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta}.$$

Here $\mathcal{H}$ is Hamiltonian function, the total energy of the fluid.

Hamiltonian



$$
\begin{aligned}
H &= \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int\{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \\
&+ \frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \ldots
\end{aligned}
$$

$$
\hat{k} = \sqrt{-\frac{\partial^2}{\partial x^2}}
$$

Normal variables a_k

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \quad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \dots$$

$$\begin{aligned} \mathcal{H}_2 &= \int \omega_k |a_k|^2 \\ \mathcal{H}_3 &= \mathcal{H}_3(a_k, a_k^*) - \text{third power} \\ \mathcal{H}_4 &= \mathcal{H}_4(a_k, a_k^*) - \text{fourth power} \end{aligned}$$

a_k satisfies the equation $\quad \dfrac{\partial a_k}{\partial t} + i\dfrac{\delta H}{\delta a_k^*} = 0$

For gravity waves three-wave processes are forbidden

$$\vec{k} = \vec{k}_1 + \vec{k}_2$$
$$\omega_k = \omega_{k_1} + \omega_{k_2}$$
$$\omega_k = \sqrt{g|k|}$$

$$a_k \Rightarrow b_k$$

Canonical transformation excludes cubic terms. After transformation b_k satisfies the equation:

$$i\dot{b}_k = \omega_k b_k + \int \mathbf{T}_{kk_1}^{k_2k_3} \underline{b_{k_1}^* b_{k_2} b_{k_3}} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

Miracle #1

k_i are one-dimensional vectors. Resonant conditions

$$k + k_1 = k_3 + k_3$$
$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}$$

if $k_1, k_2, k_3 > 0,\ \ k < 0, \Rightarrow T_{kk_1}^{k_2k_3} \equiv 0!$

In other words $\mathbf{T}_{k_2k_3}^{kk_1} = \theta(kk_1k_2k_3)\mathbf{W}_{k_2k_3}^{kk_1}$

Let all $k_i > 0$. Then

$$\mathbf{T}_{k_2 k_3}^{k k_1} = \frac{(k k_1 k_2 k_3)^{\frac{1}{4}}}{4\pi} \left[(k k_1)^{\frac{1}{2}} + (k_2 k_3)^{\frac{1}{2}} \right] \mathrm{min}(k, k_1, k_2, k_3) \theta(k k_1 k_2 k_3)$$

One more canonical transformation makes possible to replace

$$\mathbf{T}_{k k_1}^{k_2 k_3} \Rightarrow \tilde{T}_{k k_1}^{k_2 k_3}$$

$$\tilde{T}_{k k_1}^{k_2 k_3} = \frac{(k k_1 k_2 k_3)^{\frac{1}{2}}}{2\pi} \mathrm{min}(k, k_1, k_2, k_3) \theta(k k_1 k_2 k_3).$$

or

$$\begin{aligned} \tilde{T}_{k k_1}^{k_2 k_3} &= \theta(k k_1 k_2 k_3) \frac{(k k_1 k_2 k_3)^{\frac{1}{2}}}{8\pi} (k + k_1 + k_2 + k_3 - \\ &- |k - k_2| - |k - k_3| - |k_1 - k_2| - |k_1 - k_3|) \end{aligned}$$

$$c_k = k^{\frac{1}{2}}\theta k b_k$$

$$\frac{\partial c}{\partial t} + i\hat{\omega} c - i\hat{P}^{+}\frac{\partial}{\partial x}\left(|c|^2\frac{\partial c}{\partial x}\right) = \hat{P}^{+}\frac{\partial}{\partial x}(\mathcal{U}c)$$

this is "super compact" equation

Breather is the localized solution of the following type:

$$c(x,t) = C(x - Vt)e^{i(k_0 x - \omega_0 t)} \quad or \quad c_k = e^{i(\Omega + Vk)t}\phi_k$$

where ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k)\phi_k = \frac{1}{2}\int T_{kk_1}^{k_2 k_3}\phi_{k_1}^*\phi_{k_2}\phi_{k_3}\delta_{k+k_1-k_2-k_3}dk_1 dk_2 dk_3$$

Euler equation in conformal variables

These equations minimize the action

$$S = \int L dt, \qquad L = \int_{-\infty}^{\infty} \psi \eta_t dx - \mathcal{H}.$$

Starting from this point let us forget for a while about hydrodynamics, and consider more general case. Namely, let's think of $\mathcal{H}$ as some arbitrary functional of ψ and η.

Let $z(w,t)$ be the conformal mapping of the domain, bounded by the curve $\eta(x,t)$ to the lower half-plane of w

$$w = u + iv, \qquad -\infty < u < \infty, \quad -\infty < v < 0$$

We introduce two functions analytic in the lower half-plane

$$z = x + iy = z(w)$$

$$\Phi = \Psi + i\hat{H}\Psi$$

These complex-valued functions are analytic in the lower half-plane $v \leq\ 0$.

"Implicit" equations of motion can be rewritten as follows:

$$z_t \bar{z}_u - \bar{z}_t z_u = -\Phi_u + \bar{\Phi}_u$$

$$\Psi_t z_u - \Psi_u z_t + \frac{1}{2}\frac{\bar{\Phi}_u^2}{\bar{z}_u} = 0$$

$$\Psi = \frac{1}{2}(\Phi + \bar{\Phi})$$

Self-similar compressed fluid

$$\eta \equiv 0$$

$$\Phi(x, y, t) = \frac{1}{2}\frac{1}{t - t_0}(x^2 - y^2)$$

$$P = -\frac{y^2}{(t - t_0)^2} \qquad P = 0, y = 0$$

In conformal variables

$$z_0 = tu \qquad \Phi_0 = \frac{1}{2}tu^2$$

Then equations for the shape of self-similar solutions are satisfied. Let us study perturbation of this solution

$$z \to ut + z \qquad \Phi \to \frac{1}{2}u^2 t + \Phi$$

Equations for finite perturbations of the self-similar solutions read

$$tz_t - uz_u + \Phi_u = P^-(\bar{z}_t z_u - z_t \bar{z}_u)$$

$$P^-\left\{\frac{u}{2}(uz_u - \Phi_u) + t(\frac{1}{2}\Phi_t - uz_t) + \Psi_t z_u - \Psi_u z_t\right\} = 0$$

Miracle # 2

These equations are satisfied if

$$z = \alpha(u) \qquad \Phi = \Phi_0(u) = \partial^{-1} u\alpha(u)$$

$\alpha(u)$ is an arbitrary! function analytic in the lower half-plane

$$\alpha(w) \to 0 \qquad Im w \to -\infty$$

Let

$$\alpha = \frac{A}{u + ia} \qquad A, a - real\ \ constants, u > 0$$

Shape of the surface is presented in the parametric form

$$x = u + \frac{Aut}{u^2 + a^2t^2} \qquad y = -\frac{aAt^2}{u^2 + a^2t^2}$$

$$\frac{\partial x}{\partial u} \to 1 \qquad at \qquad t \to \pm\infty$$

The solution describes:

1. Formation of bubbles (if $A > 0$)
2. Formation of droplets (if $A < 0$)

The face of surface is symmetric

Miracle # 3

Let us look for solution of the above equations in the form

$$z = \alpha(u) + \frac{1}{t}\, z_1(u) + \frac{1}{t^2}\, z_2(u) + \cdots$$

$$\Phi = \Phi_0(u) + \frac{1}{t}\, \Phi_1(u) + \frac{1}{t^2}\, \Phi_2(u) + \cdots$$

Now again $z_1(u)$ and $\alpha(u)$ are arbitrary functions analytic in the lower half-plane

$$\Phi_1(u) = u\, z_1(u)$$

$$u\, z_2(u) = -P^- \left(\bar{z}_1 \alpha_u - z_1\, \bar{\alpha}_u\right) \text{ (and so on)}$$

The system is integrable! A general solution depends on two arbitrary functions $z_1(u)$ and $\alpha(u)$.

"Additional" motion constants

Is the system of equation for Z, Φ integrable if the boundary conditions are "natural": $Z \to w,\ \Phi_u \to 0$ at $|w| \to \infty$

$$\text{If } \quad Z_u = \sum_{n=1}^{N} \frac{q_n}{w - a_n(t)} + \tilde{z}_u, \quad Im(a_n) > 0$$

$$\Phi_u = \sum_{n=1}^{N} \frac{k_n}{w - a_n(t)} + \tilde{\Phi}_u$$

Poles in Z_u, Φ_u are persistent and

$$\frac{dq_n}{dt} = 0 \ \frac{dk_n}{dt} = -gq_n \ k_n = -gq_n t + k_n^{(0)}$$

$q_n, k_n^{(0)}$ are "additional motion constant

Moreover, for any circle Γ (not including branch point)

$$\int_\Gamma Z_u dw = I_n \qquad \int_\Gamma \Phi_u dw = J_n$$

$$\frac{dI_n}{dt} = 0 \qquad \frac{dJ_n}{dt} = -gI_n$$

Is this system of integrals complete - open question.

Zeroes of Z_u, Φ_u are not persistent and turn to cuts.

Hence $\tilde{Z}_u \neq 0, \tilde{\Phi}_u \neq 0$.

Equations for Z_u, Φ_u have no persistent rational solutions.

Giant Breather (in the framework of Super Compact equation)

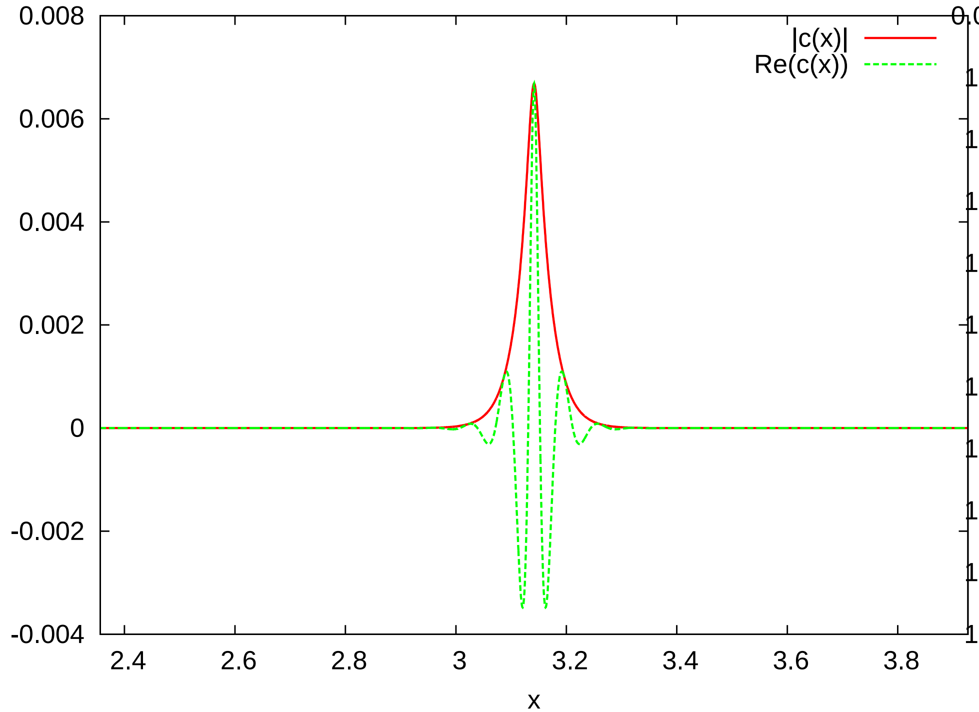


Figure 1: $|c(x)|$ (red curve)

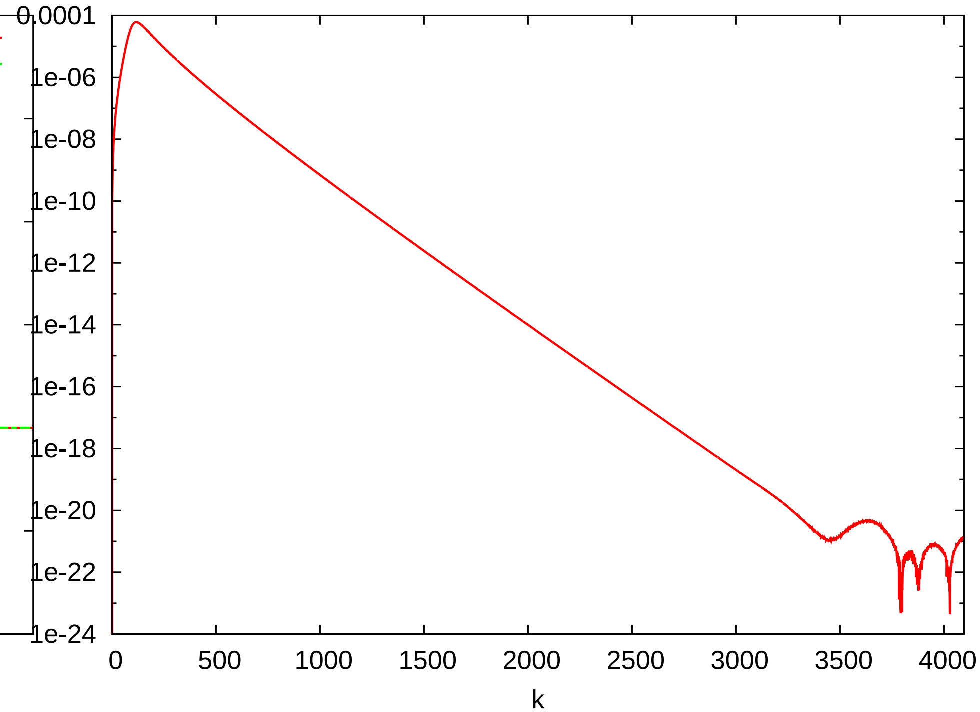
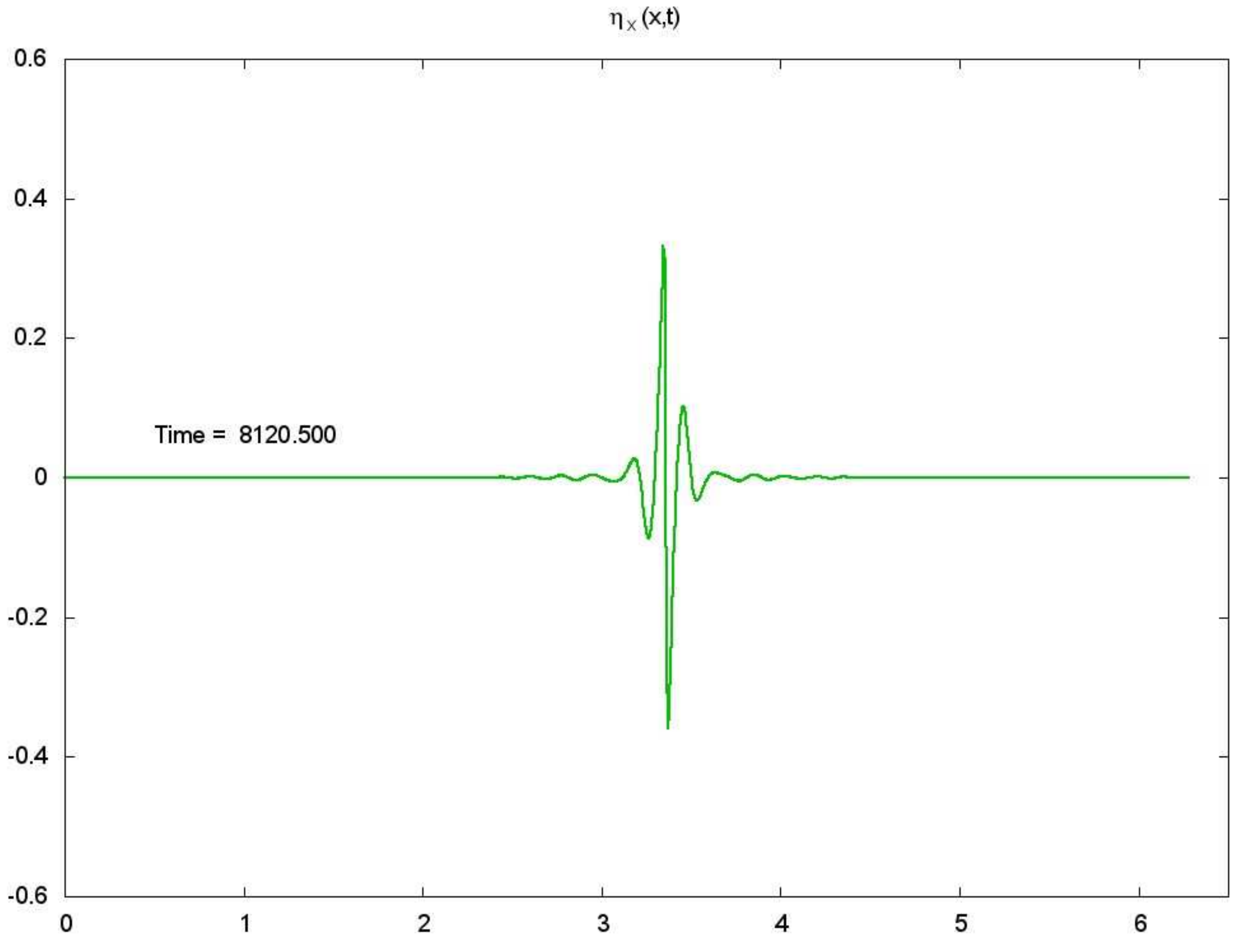


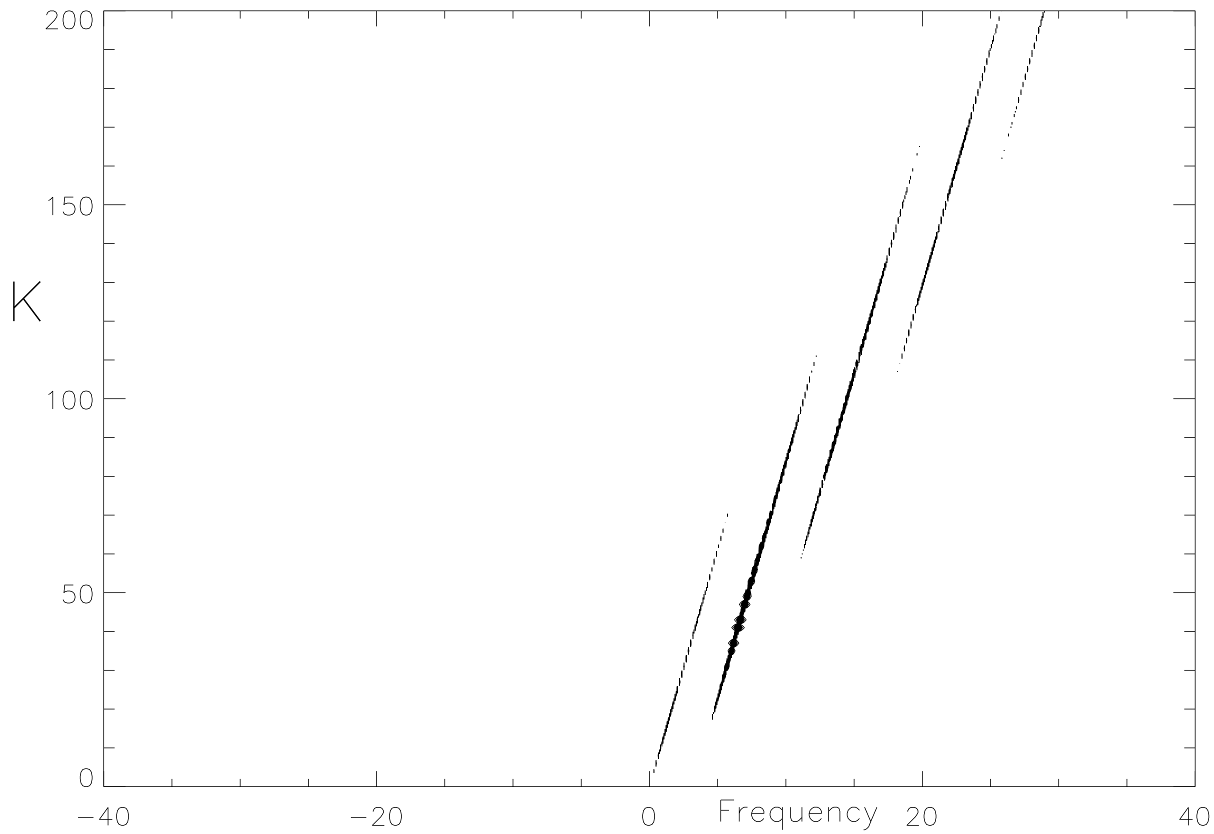
Figure 2: Spectrum $|c(k)|$)

Breather in the fully nonlinear (exact) equations (steepness).



It lives forever. This is the **Miracle #4**

$k - \omega$ spectrum of breather in the fully nonlinear equations.



No waves with negative frequency.

References

1. V.E. Zakharov, A.I. Dyachenko, Free-Surface Hydrodynamics, in the conformal variables arXiv preprint arXiv:1206.2046 (2012)

2. V.E. Zakharov, A.I. Dyachenko, S.A. Dyachenko, On additional motion constants in equations of deep water (paper in preparation)