Unresolved problems in the theory of integrable systems

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In spite of enormous success of the theory of integrable systems, at least three important problems are not resolved yet or are resolved only partly. They are the following:

1. The IST in the case of arbitrary bounded initial data.

2. The statistical description of the systems integrable by the IST. Albeit, the development of the theory of integrable turbulence.

3. Integrability of the deep water equations.

These three problems will be discussed in the talk.

Bounded non-vanishing solutions of the KdV equation

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The Korteweg-de Vries equation

The KdV equation on $u(x, t)$:

$$u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}.$$

Major open problem: For what classes of initial data can we solve the initial value problem for KdV by the use of the Inverse Scattering Transform or by other analytical methods?

Spectral theory of L and the initial value problem for KdV

To solve the initial value problem for KdV, we need to study the spectral theory of the one-dimensional Schrödinger operator L:

$$L\psi = [-\partial_x^2 + u(x)]\psi = E\psi, \quad \psi \text{ bounded.}$$

There are two important classes of potentials $u(x)$ for which the spectral theory of L is well-understood, and the corresponding initial value problem has an effective solution:

If $u(x)$ vanishes sufficiently fast as $x \to \pm\infty$, we can solve the initial value problem for KdV by using the *inverse scattering transform* (IST).

If $u(x)$ is periodic, we can approximate it and solve the initial value problem by using *finite-gap potentials*.

Motivating question. What is the relationship between the IST and finite-gap solutions?

$u(x)$ rapidly vanishing: scattering data

Suppose that $u(x)$ rapidly vanishes at infinity:

$$u(x) = O(1/x^{2+\varepsilon}), \quad x \to \pm\infty.$$

For $E = k^2 \geq 0$, the solution space has dimension 2, so there is a solution

$$\psi(x,k) = \begin{cases} e^{-ikx} + c(k)e^{ikx} + o(1) & \text{as } x \to +\infty, \\ d(k)e^{-ikx} + o(1) & \text{as } x \to -\infty. \end{cases}$$

For finitely many negative $E = -\kappa_n^2$, $n = 1, \ldots, N$, there is one solution:

$$\psi_n(x) = \begin{cases} e^{\kappa_n x}(1 + o(1)) & \text{as } x \to -\infty, \\ e^{-\kappa_n x}(b_n + o(1)) & \text{as } x \to \infty. \end{cases}$$

The set $s = \{c(k), \kappa_n, b_n\}$ is the *scattering data* of the potential $u(x)$.

KdV equations and the inverse scattering transform

If $u(x,t)$ satisfies KdV, then the spectral data $s(t)$ evolves trivially:

$$c(k,t) = c(k)e^{8ik^3t}, \quad \kappa_n(t) = \kappa_n, \quad b_n(t) = b_n e^{8\kappa_n^3 t}.$$

We can solve the initial value problem for KdV for vanishing $u(x)$:

$$u(x,0) \to s(0) \to s(t) \to u(x,t).$$

Introduce the function $F(x,t)$, where M_n is the L_2-norm $\psi_n(x)$.

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k,t)e^{ikx}dk + \sum_{n=1}^{N} M_n^2 e^{-\kappa_n x},$$

where the M_n are the L_2-norms of the eigenfunctions $\psi_n(x)$.

Solve the Marchenko equation for $K(x,y,t)$:

$$K(x,y,t) + F(x+y,t) + \int_x^{\infty} K(x,z,t)F(z+y,t)dz = 0.$$

Find the potential

$$u(x,t) = -\partial_x K(x,x,t).$$

Bargmann potentials and N-soliton solutions of KdV

The Marchenko equation can be solved explicitly in the important case $c(k) = 0$.

If $s = \{0, \kappa_n, b_n\}$, $n = 1, \ldots, N$, then $u(x)$ is a *reflectionless Bargmann potential* and $u(x,t)$ is an *N-soliton solution* of KdV.

For $N = 1$ we get a traveling solitary wave:

$$-u(x,t) = \frac{2\kappa^2}{\cosh^2 \kappa(x - 4\kappa^2 t - x_0)}.$$

In general we have N interacting solitary waves, given by the Bargmann formula

$$-u(x,t) = 2\partial_x^2 \ln \det |M_{nm}|,$$

$$M_{nm} = \delta_{nm} + c_n e^{8\kappa_n^3 t} \frac{e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m}, \quad c_n = \frac{b_n}{ia'(i\kappa_n)} > 0, \quad a(k) = \prod_{n=1}^{N} \frac{k - i\kappa_n}{k + i\kappa_n}.$$

$u(x)$ periodic: finite-gap theory

Suppose that $u(x)$ is periodic:

$$u(x+T) = u(x).$$

The spectrum of the Schrödinger operator L is described by Bloch–Floquet theory consists of an infinite sequence of closed intervals

$$\mathcal{S} = [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup [\lambda_5, \lambda_6] \cup \cdots, \quad \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

For each $E \in \mathcal{S}$, there is a two-dimensional space of solutions (one-dimensional at boundary points λ_i).

The eigenfunction $\psi(x, k)$ is defined on the *spectral curve* C: a hyperelliptic Riemann surface of infinite genus that is a double cover of the complex plane branched over the points $\lambda_1, \lambda_2, \ldots$

Finite-gap potentials

For an L^2-dense subset of periodic potentials, the spectrum has only finitely many gaps

$$S = [\lambda_1, \lambda_2] \cup \cdots \cup [\lambda_{2g-2}, \lambda_{2g-1}] \cup [\lambda_{2g}, \infty)$$

The spectral curve C is an algebraic Riemann surface of genus g.

The eigenfunction $\psi(x, k)$ has a pole divisor D of degree g on C.

$\psi(x, k)$ and $u(x)$ can be reconstructed from C and D.

If $u(x, t)$ satisfies KdV, then C does not depend on t, while D evolves linearly on the Jacobian variety $\mathrm{Jac}(C)$. The solution is given by the Matveev–Its formula

$$u(x, t) = 2\partial_x^2 \ln \theta(xU + tV + Z) + c,$$

where θ is the theta function of $\mathrm{Jac}(C)$.

For generic spectral data, this solution is quasi-periodic in x and t.

IST and finite-gap solutions

What is the relationship between the IST and finite-gap solutions?

Mumford: degenerating the spectral curve to a rational nodal curve reduces N-gap solutions to N-soliton solutions.

Idea. View finite-gap solutions as limits of soliton solutions as $N \to \infty$.

Lundina, Marchenko: Proved that periodic finite-gap solutions are contained in a suitable closure of the set of N-soliton solutions (no effective formulas).

Motivation: Fourier transform vs. d'Alembert's formula

There are two approaches to the wave equation

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty.$$

For initial data $u(x,0) = A(x)$, $u_x(x,0) = B(x)$, we find their Fourier transforms, apply time evolution, and then find the inverse Fourier transform.

Alternatively we can use d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[A(x-t) + A(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} B(s)ds.$$

The formula is local in x and t.

The IST is a nonlinear version of the Fourier transform.

Our method (we call it the dressing method) can be seen as a nonlinear version of d'Alembert's formula.

Analytic properties of ψ and χ for Bargmann potentials

In the Schrödinger equation substitute $\psi(x,k) = \chi(x,k)e^{-ikx}$:

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \quad \chi(x,k) \to 1 \text{ as } |k| \to \infty.$$

We extend χ to an analytic function in the complex k-plane.
We consider a $\overline{\partial}$-problem on the complex k-plane of the following kind:

$$\frac{\partial \chi}{\partial \overline{k}} = ie^{2ikx}T(k)\chi(x,-k).$$

Here $T(k)$ is a compactly supported distribution called the *dressing function* of the $\overline{\partial}$-problem.

$$\chi(x,k) = 1 + i\sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}.$$

The $\chi_n(x)$ and $u(x)$ are determined by the system

$$\chi_n(x) = c_n\chi(x,-i\kappa_n)e^{-2\kappa_n x}, \quad u(x) = 2\frac{d}{dx}\sum_{n=1}^{N}\chi_n(x)$$

Naive limit $N \to \infty$

Krichever, 1980s: define the limit $N \to \infty$ by allowing the poles of χ to coalesce into a jump along the negative imaginary axis.

The function χ then satisfies a singular integral equation, and its approximations by Riemann sums produce N-soliton solutions.

The resulting potentials $u(x)$ are bounded as $x \to -\infty$ but are decreasing as $x \to +\infty$.

We drop the physical assumption that there are poles only along the negative part of the imaginary axis.

The IST from the dressing method

Theorem (2014, Z., Zakharov)

Let $\kappa_1, \ldots, \kappa_N$ and $c_1, \ldots, c_n$ be nonzero real numbers satisfying $\kappa_m \neq \pm\kappa_n$ for all $m \neq n$, $c_n/\kappa_n > 0$ for all n. There is a unique rational function χ satisfying the following system:

$$\chi(x,k) = 1 + i\sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}, \quad \chi_n(x) = c_n \chi(x, -i\kappa_n) e^{-2\kappa_n x}.$$

The corresponding potential $u(x)$ is a reflectionless Bargmann potential having the finite discrete spectrum $\{-\kappa_1^2, \ldots, -\kappa_N^2\}$. Furthermore, for each n, replacing

$$\widetilde{\kappa}_i = \begin{cases} \kappa_i, & i \neq n, \\ -\kappa_n, & i = n, \end{cases} \qquad \widetilde{c}_i = \begin{cases} \left(\dfrac{\kappa_i - \kappa_n}{\kappa_i + \kappa_n}\right)^2 c_i, & i \neq n, \\ -4\pi^2\kappa_n^2/c_n, & i = n, \end{cases}$$

does not change the potential $u(x)$.

The dressing method and the symmetric contour problem

Theorem (Z., Zakharov, in progress)

Let $0 < a < b$, let R_1 and R_2 be two positive Hölder functions on $[a, b]$. Then there is a unique function χ, analytic on the k-plane away from two cuts $[ia, ib]$ and $[-ib, -ia]$ on the imaginary axis, satisfying $\chi \to 1$ as $|k| \to \infty$, which satisfies the following contour problem for $p \in [a, b]$.

$$\chi^+(x, ip) - \chi^-(x, ip) = iR_1(p)e^{-2px}[\chi^+(x, -ip) + \chi^-(x, -ip)],$$

$$\chi^+(x, -ip) - \chi^-(x, -ip) = -iR_2(p)e^{2px}[\chi^+(x, ip) + \chi^-(x, ip)].$$

The corresponding potential $u(x)$ of the Schrödinger operator

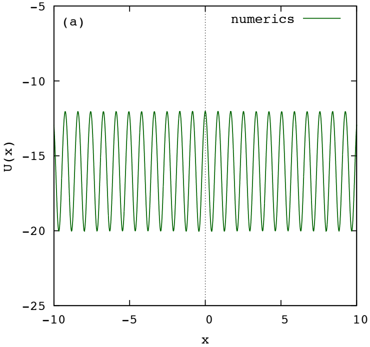
$$u(x) = 2\partial_x \chi_0(x), \quad \chi(x, k) = 1 + \frac{i\chi_0(x)}{k} + O(k^{-2})$$

is bounded as $x \to \pm\infty$ and has the spectrum $[-b^2, -a^2] \cup [0, \infty)$.

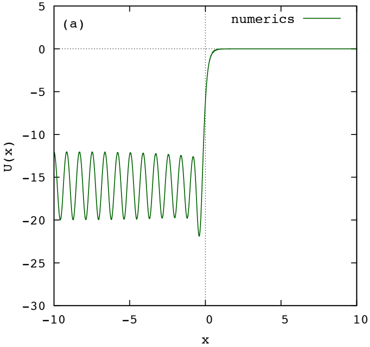
Adding time dependence corresponds to replacing e^{2px} with e^{2px+8p^3t}.

Numerical simulations for constant R_1 and R_2

We can approximately solve the Riemann–Hilbert problem using N-soliton solutions. We only consider constant R_1 and R_2 on $[a, b] = [2, 4]$.



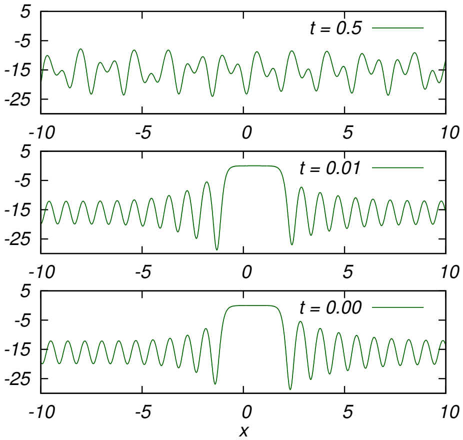
$R_1 = 1, \quad R_2 = 1$



$R_1 = 1, \quad R_2 = 0$

Numerical simulations for constant R_1 and R_2

We can approximately solve the Riemann–Hilbert problem using N-soliton solutions. We only consider constants $R_1 = 10^{-3}, \quad R_2 = 10^{-6}$ on $[a, b] = [2, 4]$. Evolution due to the KdV equation.



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