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On the Growth of Sobolev Norms in compact setting

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joint work with

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Plan of the Talk

- Introduction and Cauchy Theory: **local and global**.
- The problem of the **Growth of Sobolev norms**.
- The method of **Modified Energies**.
- Applications to **NLS in 2d**.
- Applications to **cubic NLS in 3d**.
- Applications to the **harmonic oscillator**.

Introduction to the Problem

Let us consider the following Cauchy problems:

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t,x) \in \mathbb{R} \times M^d \\ u(0,x) = \varphi(x) \in H^m \end{cases}$$

where

- (M^d, g) is a compact d-dimensional Riemannian manifold;
- Δ_g is the Laplace Beltrami operator;
- m the regularity of the initial data.

Conserved Quantity

Since the nonlinearity is **defocusing** we have the following **positive energy** which is preserved along the flow:

$$E(u(t,x)) = const$$

where

$$E(u) = \frac{1}{2}\|u\|^2_{H^1(M^d)} + \frac{1}{p+1}\|u\|^4_{L^4(M^d)};$$

moreover

$$\|u(t,x)\|_{L^2(M^d)} = const$$

Local and Global Cauchy Theory

- $2d$: the Cauchy problem is Locally Well Posed in $H^1(M^2)$;

- $3d$: the Cauchy problem is Locally Well Posed in $H^{1+\epsilon}(M^3)$;

- $2d$: it is easy to globalize the solution thanks to the conservation law;

- $3d$: the globalization argument is more involved (see Burq-Gérard-Tzvetkov).

Cheap Cauchy Theory

One can get some cheap results by using the Sobolev embedding

$$H^{d/2+\epsilon}(M^d) \subset L^\infty(M^d)$$

- $2d$: the Cauchy problem is Locally Well Posed in $H^{1+\epsilon}(M^2)$ and the solution leaves in $C((0,T); H^{1+\epsilon}(M^2))$

- $3d$: the Cauchy problem is Locally Well Posed in $H^{3/2+\epsilon}(M^3)$ for **cubic nonlinearity**, and the solution leaves in $C((0,T); H^{3/2+\epsilon}(M^2))$.

Dispersion and Strichartz Estimates

- **On a generic manifold** (M^d, g) **we have for free waves**:

$$\|e^{it\Delta_g}\varphi\|_{L^p((0,1);L^q(M^d))} \leq C\|\varphi\|_{H^{1/p}(M^d)}$$

where

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad p \geq 2, \quad (p,d) \neq (2,2)$$

(see Burq-Gérard-Tzvetkov and Staffilani-Tataru).

- **Notice that for** $d = 2$ **we get "almost" this estimate**

$$\|e^{it\Delta_g}\varphi\|_{L^2((0,1);L^\infty(M^2))} \leq C\|\varphi\|_{H^{1/2}(M^2)}$$

that compared with Sobolev embedding provides a gain of 1/2 derivative!

L.W.P. and G.W.P. for NLS in $H^1(M^2)$

NLS is the l.w.p. (and hence g.w.p.) in 2d with initial datum in $H^1(M^2)$.

- The solutions leave in $C((0,T),H^1(M^2)) \cap L^p((0,T);L^q(M^2))$.

What about the growth of $H^m(M^2)$ for $m > 1$?

Following Bourgain one can ask the following questions:

QUESTION 1: what can we say about the growth of

$$\|u(t)\|_{H^m(M^2)}$$

for $m > 1$ as $t \to \infty$ where $u(t, x)$ solves NLS on a compact manifold?

QUESTION 2: does it exists at least one solution of NLS such that

$$\|u(t)\|_{H^m(M^2)}$$

for $m > 1$ is unbounded as $t \to \infty$?

Some references

Bourgain, Staffilani, Sohinger, Colliander-Keel-Staffilani-Tao-Takaoka, Colliander- Kwon-Oh, Gérard-Grellier, Guardia-Kaloshin, Hani, Hani-Pausader-Tzvetkov-V., Haus-Procesi-Guardia, Pocovnicu, Wang, Zhong, Xu, Thirouin, Deng-Germain etc. etc.

The case: $M = \mathbb{R}^d$

- In the case $d = 1$ there is not growth of higher order Sobolev norm as a consequence of the **IST by Zakharov-Shabat**;

- For $d \geq 2$ the question can be settled by using the **Nonlinear Scattering Theory**:

  For every nonlinearity $p \geq 2 + 4/d$ *and for every* $\varphi \in H^m(\mathbb{R}^d)$ *there exist* $\varphi_\pm \in H^m(\mathbb{R}^d)$ *such that:*

$$\|u(t,x) - e^{it\Delta}\varphi_\pm\|_{H^m(\mathbb{R}^d)} \to 0 \text{ as } t \to \pm\infty.$$

  In particular since $\|e^{it\Delta}\varphi_\pm\|_{H^m(\mathbb{R}^2)} = \|\varphi_\pm\|_{H^m(\mathbb{R}^2)}$ **in the Euclidean setting there is not growth of higher order Sobolev norm**.

Cheap Growth on M^2: Exponential Growth

- To prove **exponential growth** of H^m is not complicated, once a **nice local Cauchy theory in H^1 is available**.

- In general along with the well–posedness result in $H^1(M^2)$ one can deduce, via elementary estimates, a bound of the type

$$\|u(t+\tau)\|_{H^m(M^2)} \leq C\|u(t)\|_{H^m(M^2)}$$

where $\tau = \tau(\|\varphi\|_{H^1})$. An elementary iteration gives exponential bound:

$$\|u(t,x)\|_{H^m} \leq C \exp Ct.$$

Stronger Result on M^2: Polynomial Growth of $H^s(M^2)$

- Following Bourgain's work, one can prove even more than exponential growth. In fact the higher order Sobolev norms of solutions to **cubic NLS on $\mathbb{T}^2$ have at most a polynomial growth**.

- The method pioneered by Bourgain is based on smoothing effect related with the $X^{s,b}$ spaces, namely

$$\|w(t,x)\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^b \tilde{w}(\tau,\xi)\|_{L^2_{\tau,\xi}} = \|e^{it\Delta} w(t,x)\|_{H^b_t H^s_x}$$

(roughly speaking one exploits also regularity in time and not only in space: this is very useful to solve PDEs at low regularity and also PDEs involving derivatives in the nonlinearity, e.g. KdV, Benjamin-Ono etc etc)

- More precisely the key point in the Bourgain approach is the estimate

$$\|u(t+\tau)\|_{H^m}^2 - \|u(t)\|_{H^m}^2 \leq C\|u(t)\|_{H^m}^{2-\epsilon}$$

where $\tau = \tau(\|\varphi\|_{H^1}) > 0$. Then we define $\alpha_n = \|u(\tau n)\|_{H^m}^2$ and we get

$$\alpha_{n+1} \leq \alpha_n + C\alpha_n^{1-\epsilon}$$

which in turn implies by a simple iteration argument

$$\alpha_n \lesssim n^{\frac{1}{\epsilon}}.$$

The key point in the Bourgain approach is the following computation (consider for simplicity H^2 and cubic NLS) **aimed to mimic the conservation of L^2 at higher order level**:

$$i\partial_t u + \Delta u = u|u|^2 \Rightarrow i\partial_t(\Delta u) + \Delta(\Delta u) = \Delta(u|u|^2)$$

multiply the equation by $\Delta \bar{u}$ and consider the imaginary part:

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 = \text{Im} \int \Delta \bar{u} \Delta(u|u|^2) \sim 2 \int (\Delta \bar{u})^2 u^2 + l.o.t.$$

and hence

$$\|u(t_1)\|_{H^2}^2 - \|u(t_2)\|_{H^2}^2 \sim \text{Im} \int_{t_1}^{t_2} (\Delta \bar{u})^2 \bar{u}^2 + l.o.t.$$

The idea of Bourgain it to estimate (the most dangerous term involving the square of $\Delta\bar{u}$) as follows

$$|\mathrm{Im} \int_{t_1}^{t_2} (\Delta\bar{u})^2 u^2| \lesssim \|u\|^{2-\gamma}_{X^{2,b}_{(t_1,t_2)}}$$

by exploiting the derivatives in time provided by the $X^{s,b}$ spaces... it is very technical step. In order to conclude it is necessary to estimate

$$\|u\|_{X^{2,b}_{(t_1,t_2)}} \lesssim \|u(t_1)\|_{H^2}$$

with a time interval (t_1, t_2) whose size is uniform for every $t_1 \in \mathbb{R}$. **In particular in order this approach to be successful it is necessary to solve in H^1 the Cauchy problem, since H^1 is the unique a-priori conserved quantity.**

Our Result on the Growth of $H^m(M^2)$

We assume that the Riemanian manifold (M^2, g) satisfies:

$$\|e^{it\Delta_g}\varphi\|_{L^4((0,1)\times M^2)} \lesssim \|\varphi\|_{H^{s_0}(M^2)}.$$

Following Staffilani-Tataru and Burq-Gérard-Tzvetkov it is true for every compact manifold with $s_0 = \frac{1}{4}$.

Theorem 1

Let (M^2, g) satisfies the condition above and let $u(t,x)$ be solution to NLS on M^2 with nonlinearity $u|u|^{p-1}$, $p = 2k+1$, then we get

$$\|u(t,x)\|_{H^m(M^2)} \lesssim t^{\frac{m-1}{1-2s_0}+\epsilon}, \ \forall \epsilon > 0, m \in \mathbb{N}.$$

- The previous theorem for cubic NLS on a generic M^2 has been obtained by Zhong. Our proof is different and as far as we can see it works in a more general context.

- Our argument is not based on the $X^{s,b}$ spaces, but is more elementary and **based essentially on integration by parts and Strichartz estimates**.

- **In our approach we never use the fact that the Cauchy problem is l.w.p. in $H^1(M^2)$.**

The Modified Energy Associated with H^2

Consider the following energy:

$$\mathcal{E}_2(u) = \|\Delta_g u\|_{L^2}^2 - 2\text{Re}\int_{M^2} \Delta_g u u|u|^2 - \frac{1}{2}\int_{M^2} |\nabla_g |u|^2|^2 |u|.$$

Then we have

$$\frac{d}{dt}\mathcal{E}_2(u) = -2\text{Im}\int_{M^2} (\nabla_g u, u\nabla_g |u|^4) + 2\int_{M^2} |\nabla_g u|^2 \partial_t |u|^2.$$

Estimate of $|\frac{d}{dt}\mathcal{E}_2(u(t,x))|$

Since $\mathcal{E}_2(u) \sim \|u\|_{H^2}^2$ then we have roughly after integration in dt

$$\|u(T,x)\|_{H^2}^2 - \|u(0,x)\|_{H^2}^2 \lesssim \int_0^T |\frac{d}{dt}\mathcal{E}_2(u(s))|ds$$

The typical term on the r.h.s. can be controlled as follows

$$\int_0^T \int_{M^2} |\nabla_g u|^2 |\partial_t u||u| \lesssim \|\partial_t u\|_{L_T^\infty L^2} \|u\|_{L_T^\infty L^\infty} \int_0^T \|u\|_{W^{1,4}}^2$$

$$\lesssim \|\Delta_g u\|_{L_T^\infty L^2} \|u\|_{L_T^\infty L^\infty} \int_0^T \|u\|_{W^{1,4}}^2$$

$$\lesssim \sqrt{T} \|u\|_{L_T^\infty H^2}^{1+\epsilon} \|u\|_{L^4(0,T) W^{1,4}}^2$$

$$\lesssim \sqrt{T} \|u\|_{L_T^\infty H^2}^{1+\epsilon} \|u\|_{L_T^\infty H^{1+s_0}}^2 \lesssim \sqrt{T} \|u\|_{L_T^\infty H^2}^{1+\epsilon+2s_0}$$

Cubic NLS in $3d$: exponential growth

Next we state our result in $3d$ for solutions to

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t,x) \in \mathbb{R} \times M^3 \\ u(0,x) = \varphi(x) \in H^m \end{cases}$$

Theorem 2
Let (M^3, g) be a Riemanian manifold and $p = 3$. Then for every $m \in \mathbb{N}$ and for every $u(t,x)$ solution we have:

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^3)} \leq C \exp(CT)$$

where $C = C(\|\varphi\|_{H^m}) > 0$.

- The result should be compared with the previous one by Burq-Gérard-Tzvetkov

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^3)} \leq C \exp\exp(CT)$$

Modified Energies in $3d$ for cubic NLS

Let us introduce the modified energy:

$$\mathcal{E}_2(u) = \|\Delta_g u\|_{L^2}^2 - 2\mathrm{Re}\int \Delta_g u \bar{u} |u|^2 - 2\int |\nabla_g(|u|^2)|^2$$

We have the following identity:

$$\frac{d}{dt}\mathcal{E}_2(u(t,x)) = 2\int_{M^3} |\nabla_g u|^2 \partial_t(|u|^2) - 2\mathrm{Im}\int_{M^3} (\nabla_g u, u\nabla_g(|u|^4))$$

which implies

$$\mathcal{E}_2(u(t_2,x)) - \mathcal{E}_2(u(t_2,x))$$

$$= 2\int_{t_1}^{t_2}\int_{M^3} |\nabla_g u|^2 \partial_t(|u|^2) dxdt - 2\mathrm{Im}\int_{t_1}^{t_2}\int_{M^3} (\nabla_g u, u\nabla_g(|u|^4)) dxdt$$

How to estimate $|\frac{d}{dt}\mathcal{E}_2(u(t,x))|$?

Let's deal with the most dangerous term:

$$|\int_{M^3} |\nabla_g u|^2 \partial_t(|u|^2)| \lesssim \|\partial_t u\|_{L^\infty_T L^2} \|u\|^2_{L^2_T W^{1,6}} \|u\|_{L^\infty_T L^6}$$

by using the equation and the Sobolev embedding $H^1 \subset L^6$ we get

$$... \lesssim \|u\|_{L^\infty_T H^2} \|u\|^2_{L^2_T W^{1,6}}$$

If we show that

$$\|u\|^2_{L^2_T W^{1,6}} \lesssim \|u\|_{L^\infty_T H^2}$$

then by recalling that $\mathcal{E}_2(u) \sim \|u\|^2_{H^2}$ we get

$$\|u(T)\|^2_{H^2} - \|u(0)\|^2_{H^2} \lesssim \|u\|^2_{L^\infty_T H^2}$$

then we conclude by Gronwall the exponential growth.

How to deal with $\|u\|_{L^2_T W^{1,6}}$?

We have the Strichartz estimate:

$$\|\pi_N u\|_{L^2_{(0,T)};L^6(M^3)} \leq C\|\pi_N u\|_{L^2_{(0,T)}H^{1/2}(M^3)} + \|\pi_N F\|_{L^2_{(0,T)};L^{6/5}(M^3)}$$

where π_N are the Littlewood-Paley localization operators and

$$i\partial_t u + \Delta_g u = F, \quad (t,x) \in \mathbb{R} \times M^3$$

Now we square and we get

$$\sum_N \|\pi_N u\|^2_{L^2_T W^{1,6}} \lesssim \sum_N \|\pi_N u\|^2_{L^2_T H^{3/2}} + \sum_N \|\pi_N F\|^2_{L^2_T W^{1,6/5}}$$

that implies

$$\|u\|_{L^2_T W^{1,6}} \lesssim \|u\|_{L^2_T H^{3/2}(M^3)} + \|F\|_{L^2_T W^{1,6/5}}$$

if u solves cubic NLS

$$\|u\|_{L^2_T W^{1,6}} \lesssim \|u\|_{L^2_T H^{3/2}} + \|u|u|^2\|_{L^2_T; W^{1,6/5}}$$

$$\lesssim \sqrt{T} \|u\|^{\frac{1}{2}}_{L^2_T H^1} \|u\|^{\frac{1}{2}}_{L^2_T H^2} + \sqrt{T} \|\nabla u\|_{L^\infty_T L^2} \|u\|^2_{L^\infty_T L^6}$$

and hence (by the conservation of the energy)

$$\|u\|^2_{L^2_T W^{1,6}} \lesssim T + T\|u\|_{L^\infty_T H^2}$$

The Harmonic Oscillator

Consider NLS perturbed by the potential $|x|^2$:

$$\begin{cases} i\partial_t u + Hu + u|u|^2 = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^2 \\ u(0,x) = \varphi(x) \in H^m \end{cases}$$

where, $H = -\Delta + |x|^2$.

- **We have discrete spectrum hence no global Strichartz estimates and hence no scattering**;
- The Cauchy theory is well-established since we have **Strichartz estimates local in time**;
- **What about the large-time behavior of higher order Sobolev norms?**

Modified Energy for the Harmonic Oscillator

$$\frac{1}{2}\frac{d}{dt}[\|Hu\|_{L^2}^2 + \frac{1}{4}\int |x|^2|u|^4 - |\nabla u|^2|u|^2 + \frac{1}{2}\text{Re}(\partial_{x_1}\bar{u})^2 u^2 + \frac{1}{2}\text{Re}(\partial_{x_2}\bar{u})^2 u^2]$$

$$= \int (|\nabla u|^2)\partial_t |u|^2 + \frac{1}{2}\text{Re}\int (\partial_{x_1}\bar{u})^2 \partial_t(u^2) + \frac{1}{2}\text{Re}\int (\partial_{x_2}\bar{u})^2 \partial_t(u^2)$$

In the context of the harmonic oscillator (see A. Poiret) we have the following **bilinear effect** that allows us to estimate better the r.h.s

$$\|e^{itH} f_N \cdot e^{itH} g_M\|_{L^2_{t,x}} \leq \frac{\min\{N, M\}}{\max\{N, M\}} \|f_N\|_{X^{0,b}} \|g_M\|_{X^{0,b}}$$

where $\pi_N(f_N) = f_N$ and $\pi_M(g_M) = g_M$, that implies

$$\|v_N \cdot w_M\|_{L^2_{t,x}} \leq \frac{\min\{N, M\}}{\max\{N, M\}} \|v_N\|_{X^{0,b}} \|w_M\|_{X^{0,b}}$$

where $\pi_N(v_N) = v_N$ and $\pi_M(w_M) = w_M$.

Thanks to this bilinear effect and the introduction of the modified energy we get

Theorem 3

Let $u(t,x)$ be solution to cubic NLS perturbed by the harmonic oscillator, then we have the bound

$$\|u(t,x)\|_{H^m} + \||x|^m u\|_{L^2} \leq CT^{\frac{2}{3}(m-1)+\epsilon}$$

- Recall that for NLS on $\mathbb{T}^2$ we get $\|u(t,x)\|_{H^m} \leq CT^{m-1+\epsilon}$;

- notice that ee control the momentum together with the Sbolev norm H^m;

- the result is an extension of previous one by Colliander-Delort-Kenig-Staffilani in the euclidean setting $\mathbb{R}^2$.

We integrate w.r.t. dt the expression $\frac{d}{dt}\mathcal{E}(u(t))$ and after integration we are reduced to estimate terms of the following type

$$\int\int \Delta u^0 \nabla u^1 \nabla u^2 u^3 dxdt$$

where u^i can be u or $\bar{u}$. Then by Fourier localization we get

$$\int\int \Delta u^0 \nabla u^1 \nabla u^2 u^3 dxdt = \sum_{N_0, N_1, N_2, N_3} N_0^2 u_{N_0}^0 N_1 u_{N_1}^1 N_2 u_{N_2}^2 u_{N_3}^3$$

$$\lesssim \|u^0\|_{X^{3/2+\epsilon,b}} \|u^1\|_{X^{1+\epsilon,b}} \|u^2\|_{X^{1+\epsilon,b}} \|u^3\|_{X^{1/2+\epsilon,b}}$$

and we can continue by using the l.w.p. in the Bourgain spaces, that still follows by the bilinear effect.

- A technical point is that it is not completely clear that

$$\pi_N \nabla u \sim N \pi_N u$$

since you have straight derivatives and you localize along the operator H and hence they don't commute.

- Of course no problems if we replace ∇ by $\sqrt{H}$ but this is not the case.

- Indeed one can prove

$$\|\nabla u\|_{X^{s,b}} \lesssim \|u\|_{X^{1+s,b}}$$

where $X^{s,b}$ is the Bourgain space associated with the operator H.

Thank You for Your Attention!