Construction of finite time blow up for the Wave Map System

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The wave map system

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Equivariant solutions :

$$f(t, r, \theta) = (\sin(u(r))\cos(k\phi), \sin(u(r))\sin(k\phi), \cos(u(r)))$$

$$\begin{cases} u : (0, \infty) \to \mathbb{R} \\ \partial_{tt} u - \partial_{rr} u - \frac{1}{r}\partial_r u + \frac{k^2}{2r^2}\sin(2u) = 0. \end{cases} \quad (1)$$

Adapted space :

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<u>Goal</u> : explicit solutions with a blow up at time $T < \infty$ and the regular part, and find the rate λ.

Théorème (Raphaël, Rodnianski 2010)

There exists an open set $\mathcal{O}$ of initial condition in $H^2 + Q$ such that the solution u of (WM) with an IC in $\mathcal{O}$ blows up at time $T < \infty$ with a rate $\lambda(t)$ satisfying :

$$u(t, \lambda(t)y) \underset{t \to T^+}{\to} Q(y) \text{ in } H^1_{r,loc}$$

and

$$\lambda(t) \sim \frac{T - t}{|log(T - t)|^{1/(2k-2)}} \text{ for } k \geq 2,$$

$$\lambda(t) \sim (T - t) \exp(-\sqrt{|\log(T - t)|}) \text{ fork} = 1,$$

and such that the regular part $(u^*, v^*) \in H$ verifies :

$$\lim_{t \to T} E \left(u(t, r) - Q(\frac{r}{\lambda(t)}) - u^*(r), \partial_t u(t, r) - v^*(r) \right) = 0.$$

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- To be continued...