Dispersive partial differential equations on the half-line

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Joint work with B. Erdogan and B. Gurel
For manuscripts see http://www.math.uiuc.edu/∼tzirakis/

Consider the nonlinear Schrödinger equation (NLS) on $\mathbb{R}$ or $\mathbb{T}$:

$$iu_t + u_{xx} \pm |u|^2 u = 0, \quad x \in \mathbb{R} \text{ or } \mathbb{T}, t \in \mathbb{R},$$
$$u(\cdot, 0) = g(\cdot) \in H^s.$$

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• Duhamel's formula

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• Bourgain '93: Wellposedness in $H^s(\mathbb{T})$, $s \geq 0$; using periodic $L^4_{x,t}$ Strichartz and $X^{s,b}$ spaces:

$$\|u\|_{X^{s,b}} = \left\| e^{-it\partial_{xx}} u \right\|_{H^s_x H^b_t} = \left\| \langle k \rangle^s \langle \tau + k^2 \rangle^b \, \widehat{u}(k,\tau) \right\|_{\ell^2_k L^2_\tau}.$$

Smoothing theorem for NLS, Erdogan–Tz. (MRL '13)

- Fix $s > 0$. Assume that $g \in H^s(\mathbb{T})$. Then for any $a < \min(2s, 1/2)$,

$$u(x,t) - e^{i(\partial_{xx}+P)t} g \in C^0_{t\in\mathbb{R}} H^{s+a}_{x\in\mathbb{T}},$$

where $P = \|g\|_2^2/\pi$.

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- For fixed $s > 0$, $a < \min(2s, \frac{1}{2})$, $0 < b - \frac{1}{2}$ sufficiently small :

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• Compaan '14: the analogous statement on $\mathbb{R}$.

• Kappaler–Schaad–Topalov '15: defocusing NLS on $\mathbb{T}$. For $g \in H^k$, $k \geq 1$ integer,

$$\|u - e^{iLt} g\|_{H^{k+1}} \leq C(\|g\|_{H^k}), \;\; t \in \mathbb{R}$$

where L is a linear operator depending on g nonlinearly.

Improved smoothing theorem for NLS, Erdogan–Gurel–Tz. (to appear in Indiana Math. J.)

Consider the cubic NLS on $\mathbb{T}$. For any $s > \frac{1}{4}$, and $a \leq \min(1, 2s)$ (the inequality has to be strict if the minimum is $2s$), we have

$$\|u(t) - e^{it(\partial_{xx}+P)}g\|_{C_t^0 H_x^{s+a}} \lesssim \|g\|_{H^s}^3 + \|g\|_{H^s}^5.$$

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On $\mathbb{R}$, for any $s > \frac{1}{2}$, and $a < 1$, we have $u(t) - e^{it\partial_{xx}}g \in C_t^0 H_x^{s+a}$ and

$$\|u(t) - e^{it\partial_{xx}}g\|_{H^{s+a}} \lesssim \|g\|_{H^s}(1 + \|g\|_{H^{\frac{1}{2}+}}^2) + \|u(t)\|_{H^s}\|u(t)\|_{H^{\frac{1}{2}+}}^2 + \int_0^t \|u(t')\|_{H^s}\big(\|u(t')\|_{H^{\frac{1}{2}+}}^2 + \|u(t')\|_{H^{\frac{1}{2}+}}^4\big)dt'.$$

NLS on $\mathbb{R}^+$:

We study the following initial-boundary value problem (IBVP)

$$\begin{aligned} &iu_t + u_{xx} \pm |u|^2 u = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ &u(x,0) = g(x), \quad u(0,t) = h(t). \end{aligned} \tag{1}$$

Here $g \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, $s \in [0, \frac{5}{2}) \backslash \{\frac{1}{2}, \frac{3}{2}\}$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$.

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- Kato smoothing:

$$\left\| \eta(t) e^{it\partial_{xx}} g \right\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} \lesssim \|g\|_{H^s(\mathbb{R})}.$$

To construct the solutions of (1), first consider the linear problem:

$$iv_t + v_{xx} = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \tag{2}$$
$$v(x,0) = g(x) \in H^s(\mathbb{R}^+), \quad v(0,t) = h(t) \in H^{\frac{2s+1}{4}}(\mathbb{R}^+).$$

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The solution v can be written as

$$W_0^t(g,h) = W_0^t(0, h-p) + e^{it\partial_{xx}} g_e,$$

where g_e is an H^s extension of g to $\mathbb{R}$, and $p(t) = e^{it\partial_{xx}} g_e\big|_{x=0}$, which is locally in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ by Kato smoothing.

Here $W_0^t(0,h)$ denotes the solution of (2) when $g = 0$.

Using Laplace transform (Bona–Sun–Zhang), we have $W_0^t(0,h) = W_1 h + W_2 h$, where

$$W_1 h(x,t) = \frac{1}{\pi}\int_0^\infty e^{-i\beta^2 t + i\beta x}\beta\widehat{h}(-\beta^2)d\beta,$$

$$W_2 h(x,t) = \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t - \beta x}\rho(\beta x)\beta\widehat{h}(\beta^2)d\beta,$$

$\rho(x)$: a smooth function supported on $(-2,\infty)$, $\rho(x) = 1$ for $x > 0$.

$$\widehat{h}(\xi) = \mathcal{F}\big(\chi_{(0,\infty)}h\big)(\xi) = \int_0^\infty e^{-i\xi t}h(t)dt.$$

$W_0^t(0,h)$ is well-defined for $x \in \mathbb{R}$, but satisfies linear Schrödinger for $x > 0$.

Duhamel's formula for (1).

$$u(t) = W_0^t(g, h - q)(t) + \int_0^t e^{i(t-t')\partial_{xx}} |u|^2 u \, dt', \tag{3}$$

$$q(t) = \int_0^t e^{i(t-t')\partial_{xx}} |u|^2 u \, dt' \Big|_{x=0}.$$

We solve (3) on $\mathbb{R}$. The restriction of the solution to $\mathbb{R}^+$ satisfies NLS.

Fixed point argument on $X^{s,b}$ on $\mathbb{R}$ for $b < \frac{1}{2}$, $s > 0$.

In addition, the solution is in

$$C_t^0 H_x^s \cap C_x^0 H_t^{\frac{2s+1}{4}}.$$

$X^{s,b}$ estimates ($0 \leq b < \frac{1}{2}$)

$$\left\| \eta(t) \int_0^t e^{i(t-t')\partial_{xx}} F(t') dt' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,-b}}, \quad s \in \mathbb{R},$$

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$$\|\eta(t) W_0^t(g,h)\|_{X^{s,b}} \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)}, \quad s \ge 0,$$

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$$\left\| \eta(t) \int_0^t e^{i(t-t')\partial_{xx}} F dt' \right\|_{C_x^0 H_t^{\frac{2s+1}{4}}} \lesssim \begin{cases} \|F\|_{X^{s,-b}} & 0 \leq s \leq 1/2, \\ \|F\|_{X^{\frac{1}{2}, \frac{2s-1-4b}{4}}} + \|F\|_{X^{s,-b}} & 1/2 \leq s \leq 5/2. \end{cases}$$

Proposition (Erdogan–Tz. (JFA ’16))

For fixed $0 < s < \frac{5}{2}$*, and* $0 \leq a < \min(2s, \frac{1}{2}, \frac{5}{2} - s)$*, there exists* $\epsilon > 0$ *such that for* $\frac{1}{2} - \epsilon < b < \frac{1}{2}$*, we have*

$$\big\| |u|^2 u \big\|_{X^{s+a,-b}} \lesssim \|u\|^3_{X^{s,b}},$$

$$\big\| |u|^2 u \big\|_{X^{\frac{1}{2}, \frac{2s+2a-1-4b}{4}}} \lesssim \|u\|^3_{X^{s,b}}, \quad \text{for} \quad 1/2 < s + a < 5/2.$$

- Yields the local theory for $0 < s < \frac{5}{2}$.

- Also yields the following smoothing statement:

Theorem (Erdogan–Tz. (JFA '16))

Fix $s \in (0, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, *and* $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, *with the additional compatibility condition* $g(0) = h(0)$ *for* $s > \frac{1}{2}$. *Then, for t in the local existence interval* $[0, T]$ *and* $a < \min(2s, \frac{1}{2}, \frac{5}{2} - s)$ *we have*

$$u(x,t) - W_0^t(g,h) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+).$$

Energy identities

Recall that on $\mathbb{R}$: $\|u\|_{L^2} = \|g\|_{L^2}$, and

$$E(t) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 \mp \frac{1}{4}\|u\|_{L^4}^4 = E(0).$$

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The following provide a priori bounds for the H^1 norm of the solution, bounded in the defocusing, and exponential in the focusing case.

$$\partial_t |u|^2 = -2\Im(u_x \overline{u})_x,$$

$$\partial_t(|u_x|^2 \mp \frac{1}{2}|u|^4) = 2\Re(u_x \overline{u_t})_x,$$

$$\partial_x(|u_x|^2 \pm \frac{1}{2}|u|^4) = -i\big[(u\overline{u_x})_t - (u\overline{u_t})_x\big].$$

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- Bona–Sun–Zhang '15. Solution is global in H^1 if $g, h \in H^1$.

Theorem (Erdogan–Tz. (JFA ’16))

In the case $s \in [1, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, the solution u is global and the smoothing statement holds for all times. Moreover, in the defocusing case $\|u\|_{H^s(\mathbb{R}^+)}$ grows at most polynomially, whereas in the focusing case it grows at most exponentially.

• Zakharov system on $\mathbb{R}^+$:

$$\begin{cases} iu_t + u_{xx} = nu, \quad x \in \mathbb{R}^+, \quad t \in \mathbb{R}^+, \\ n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = g(x) \in H^{s_0}(\mathbb{R}^+), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{R}^+), \quad n_t(x,0) = n_1(x) \in \hat{H}^{s_1-1}(\mathbb{R}^+), \\ u(0,t) = h(t) \in H^{\frac{2s_0+1}{4}}(\mathbb{R}^+), \;\; n(0,t) = f(t) \in H^{s_1}(\mathbb{R}^+), \end{cases}$$

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• Erdogan–Tz: Local theory and smoothing for admissible (s_0, s_1): $\frac{3}{2} > s_1 > -\frac{1}{2}$ and $\max(s_1, \frac{s_1}{2} + \frac{1}{4}) < s_0 \leq s_1 + 1$.

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• On $\mathbb{R}$ the local theory is studied in $X^{s,b}$ spaces (Bourgain, Colliander, Ginibre, Tsutsumi, Velo).

Theorem (Erdogan–Tz. Comm. PDE '17)

For any admissible pair (s_0, s_1) *the Zakharov system is locally wellposed in* $H^{s_0}(\mathbb{R}^+) \times H^{s_1}(\mathbb{R}^+)$. *Moreover, in the noncritical case when* $s_0 < s_1 + 1$, *we have the following smoothing bounds*

$$u - W_0^t(g, h) \in C_t^0 H^{s_0+a_0}([0, T] \times \mathbb{R}^+)$$
$$n - V_0^t(n_0, n_1, f) \in C_t^0 H^{s_1+a_1}([0, T] \times \mathbb{R}^+),$$

for any $a_0 < \min\left(\frac{1}{2}, s_1 + \frac{1}{2}, s_1 - s_0 + 1, \frac{5}{2} - s_0\right)$ *and* $a_1 < \min\left(s_0 - s_1, 2s_0 - s_1 - \frac{1}{2}, \frac{3}{2} - s_1\right)$. *In the case* $s_0 = s_1 + 1$, *we have the one sided smoothing*

$$n - V_0^t(n_0, n_1, f) \in C_t^0 H^{s_1+a_1}([0, T] \times \mathbb{R}),$$

for any $a_1 < \min\left(1, \frac{3}{2} - s_1\right)$.

• Derivative Schrödinger equation on $\mathbb{R}^+$:

$$\begin{cases} iu_t + u_{xx} - i(|u|^2 u)_x = 0, & x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ u(x,0) = G(x), & u(0,t) = H(t). \end{cases}$$

$\mathcal{G}_\alpha f(x) = f(x) \exp\big(-i\alpha \int_x^\infty |f(y)|^2 dy\big)$, $\alpha \in \mathbb{R}$

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If u solves the above equation, then $v = \mathcal{G}_\alpha u$ satisfies

$$\begin{cases} iv_t + v_{xx} - i(2\alpha+1)v^2\overline{v}_x - i(2\alpha+2)|v|^2 v_x + \frac{\alpha}{2}(2\alpha+1)|v|^4 v = 0 \\ v(x,0) = g(x), \quad v(0,t) = h(t), \end{cases}$$

where $g(x) = \mathcal{G}_\alpha G(x)$, and

$$h(t) = H(t) \exp\left(-i\alpha \int_0^\infty |u(y,t)|^2 dy\right) = H(t) \exp\left(-i\alpha \int_0^\infty |v(y,t)|^2 dy\right).$$

For $\alpha = -1$

$$\begin{cases} iv_t + v_{xx} + iv^2\overline{v}_x + \frac{1}{2}|v|^4 v = 0, & x, t \in \mathbb{R}^+, \\ v(x,0) = g(x), \quad v(0,t) = h(t). \end{cases}$$

Theorem

Fix $s \in (\frac{1}{2}, \frac{5}{2})$, $s \neq \frac{3}{2}$, and $a < \min(\frac{5}{2} - s, \frac{1}{4}, 2s - 1)$. Then the solution v satisfies

$$v(x,t) - W_0^t(g,h)(x) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+),$$

where T is the local existence time, and $W_0^t(g,h)$ is the solution of the corresponding linear equation.

THREE PROBLEMS

1. Local well-posedness from v to u

2. Uniqueness of strong solutions

3. Global well-posedness theory in the energy space

1. The first requires to prove a different boundary value problem since the boundary function depends on the value of the function in the interior of the domain. Need to find h of the form $e^{i\gamma(t)}H(t)$, so that the solution v with data g, h, satisfies

$$\int_0^\infty |v(y,t)|^2 dy = \gamma(t), \quad t \in [0,T].$$

Lemma

Given $G \in H^s(\mathbb{R}^+)$ and $H \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, there is a unique real valued function $\gamma \in H^{\frac{2s+1}{4}}([0,T])$ such that the solution v with data $g(x) = e^{i\int_x^\infty |G(y)|^2 dy} G(x)$, and $h(t) = e^{i\gamma(t)}H(t)$, satisfies

$$\gamma(t) = \int_0^\infty |v(y,t)|^2 dy, \ \ t \in [0,T].$$

Here $T = T(\|G\|_{H^s}, \|H\|_{H^{\frac{2s+1}{4}}})$. Moreover, γ depends on G and H continuously.

2. The second requires an iteration of the smoothing estimates

First step: Uniqueness for smooth solutions/Energy arguments

Second step: Smoother approximation and smoothing

3. The third requires small initial and boundary data

Theorem

For any $\alpha \in \mathbb{R}$, there exists an absolute constant $c > 0$ such that DNLS on half line is globally well-posed in $H^1(\mathbb{R}^+)$ provided that $\|g\|_{H^1(\mathbb{R}^+)} + \|h\|_{H^1(\mathbb{R}^+)} \leq c$.