Infinitely many conserved quantities for the cubic Gross-Pitaevskii hierarchy in 1D

Nataša Pavlović, University of Texas at Austin

Based on works with
Chen, Chen - Tzirakis, Chen - Hainzl - Seiringer,
and the recent work with
Mendelson - Nahmod - Staffilani.

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Outline

Interacting bosons

At very low temperatures dilute Bose gases are characterized by the "macroscopic occupancy of a single one-particle state", the phenomenon known as Bose-Einstein condensation.

- **The prediction** in 1920's
  Bose, Einstein
- **The first experimental realization** in 1995
  Cornell-Wieman et al, Ketterle et al
- **Proof of Bose-Einstein condensation**
  - Then (around 2000):
    Lieb-Seiringer, Lieb-Seiringer-Yngvason, Lieb-Yngvason
  - Now:
    Nam-Rougerie-Seiringer, Boccato-Brennecke-Cenatiempo-Schlein

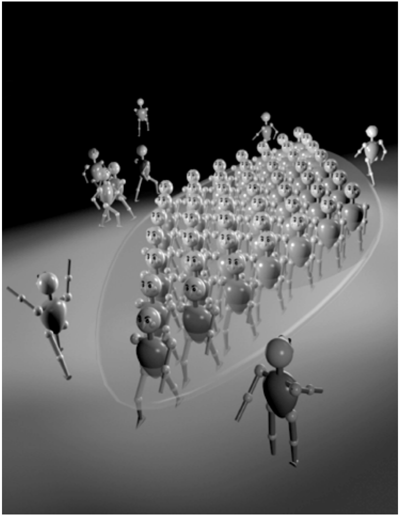

Figure : Artist's conception of particles in BEC. Cover page of the Science Magazine, Dec. 22, 1995.

Nonlinear Schrödinger equation (NLS)

The mathematical analysis of the nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u = \lambda |u|^{p-1} u \tag{1.1}$$

$$u(x,0) = u_0(x) \in H^s(\Omega^n), \; t \in \mathbb{R}, \tag{1.2}$$

where Ω^n is either the space $\mathbb{R}^n$ or the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, has been a hot topic in PDE...

Bosons and NLS

What is a connection between:

- interacting bosons

  and

- NLS?

Rigorous derivation of the NLS from quantum many body systems

- How: a few different approaches (via QFT, Math Physics or PDE)
- Then (in the late '70s and the '80s):
  - via Quantum Field Theory (*Hepp, Ginibre-Velo*)
  - via Math Physics (*Spohn*)
- Now:
  - via Quantum Field Theory (*Rodnianski-Schlein, Grillakis-Machedon-Margetis, Grillakis-Machedon, X. Chen, Pickl, Brennecke-Schlein*)
  - via Math Physics (*Fröhlich-Tsai-Yau, Bardos-Golse-Mauser, Erdös-Yau, Adami-Bardos-Golse-Teta, Elgart-Erdös-Schlein-Yau, Erdös-Schlein-Yau*)
  - via Math Physics + Dispersive PDE (*Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani, Chen-P., Chen-P.-Tzirakis, Gressman-Sohinger-Staffilani, Sohinger, X. Chen, X. Chen-Holmer, X. Chen-Smith, Chen-Hainzl-P.-Seiringer, Hong-Taliaferro-Xie, Herr-Sohinger*)

From bosons to NLS, following Erdös-Schlein-Yau [2006-07]

Step 1: From N-body Schrödinger to the (infinite) GP[1] hierarchy

Step 2: Uniqueness of solutions to the GP hierarchy

The link with the NLS: Since GP admits special factorized solutions (with each factor solving the cubic NLS), uniqueness of the GP implies that for factorized initial data, the solutions of the GP hierarchy are determined by a cubic NLS.

[1] Gross-Pitaevskii

Step 1: From N-body Schrödinger to the (infinite) GP hierarchy

The starting point is **a system of N bosons whose dynamics is generated by the Hamiltonian**

$$H_N := \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \le i < j \le N} V_N(x_i - x_j) \,, \tag{2.1}$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$, whose elements $\Psi(x_1, \dots, x_N)$ are fully symmetric with respect to permutations of the arguments x_j.

Here

$$V_N(x) = N^{d\beta} V(N^\beta x),$$

with $0 < \beta \le 1$.

When $\beta = 1$, the Hamiltonian

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\leq i<j\leq N} V_N(x_i - x_j)\,, \tag{2.2}$$

is called the Gross-Pitaevskii Hamiltonian.

- We note that physically (2.2) describes a very dilute gas, where **interactions among particles are very rare and strong**.
- This is in contrast to a mean field Hamiltonian, where each particle usually reacts with all other particles via a very weak potential.
- However thanks to the factor $\frac{1}{N}$ in front of the interaction potential, (2.2) can be formally interpreted as a mean field Hamiltonian. In particular, one can still apply to (2.2) similar mathematical methods as in the case of a mean field potential.

On the N-body Schrödinger equation

OK news:

- Since the N-body Schrödinger equation $i\partial_t\psi_N = H_N\psi_N$, is linear and the Hamiltonian H_N is self-adjoint, global well-posedness is not an issue.

Bad news:

- Qualitative and quantitative properties of the solution are hard to extract in physically relevant cases when number of particles N is very large (e.g. it varies from 10^3 for very dilute Bose-Einstein samples, to 10^{30} in stars).

Good news:

- Physicists often care about macroscopic properties of the system, which can be obtained from averaging over a large number of particles.
- Further simplifications are related to obtaining a macroscopic behavior in the limit as $N \to \infty$, with a hope that the limit will approximate properties observed in the experiments for a very large, but finite N.

Go to the GP directly

To study the limit as $N \to \infty$, one introduces:

- **the N-particle density matrix**

$$\gamma_N(t, \underline{x}_N; \underline{x}'_N) = \Psi_N(t, \underline{x}_N)\overline{\Psi_N(t, \underline{x}'_N)},$$

- **and its k-particle marginals**

$$\gamma_N^{(k)}(t, \underline{x}_k; \underline{x}'_k) = \int d\underline{x}_{N-k}\gamma_N(t, \underline{x}_k, \underline{x}_{N-k}; \underline{x}'_k, \underline{x}_{N-k}),$$

for $k = 1, \dots, N$.

Here

$$\underline{x}_k = (x_1, \dots, x_k),$$
$$\underline{x}_{N-k} = (x_{k+1}, \dots, x_N).$$

The BBGKY[2], hierarchy is given by

$$i\partial_t\gamma_N^{(k)} = -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma_N^{(k)}$$

$$(2.3) \qquad +\frac{1}{N}\sum_{1\le i<j\le k}\big(V_N(x_i-x_j)-V_N(x'_i-x'_j)\big)\,\gamma_N^{(k)}$$

$$(2.4) \qquad +\frac{N-k}{N}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\big(V_N(x_j-x_{k+1})-V_N(x'_j-x'_{k+1})\big)\,\gamma_N^{(k+1)}$$

In the limit $N\to\infty$, the sums weighted by combinatorial factors have the following size:

- In (2.3), $\frac{k^2}{N}\to 0$ for every fixed k and sufficiently small β.
- In (2.4), $\frac{N-k}{N}\to 1$ for every fixed k and $V_N(x_i-x_j)\to b_0\delta(x_i-x_j)$, with $b_0=\int dx\,V(x)$.

[2] Bogoliubov-Born-Green-Kirkwood-Yvon

The infinite Gross-Pitaevskii (GP) hierarchy

As $N \to \infty$, one obtains the infinite GP hierarchy:

$$i\partial_t \gamma^{(k)} = -\sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x_j'}) \gamma^{(k)} + b_0 \sum_{j=1}^{k} B_{j;k+1} \gamma^{(k+1)}$$

where the **"contraction operator"** is given via

$$\begin{aligned}\left(B_{j;k+1}\gamma^{(k+1)}\right)&(t, x_1, \dots, x_k; x_1', \dots, x_k') \\ &= \gamma^{(k+1)}(t, x_1, \dots, x_j, \dots, x_k, x_j; x_1', \dots, x_k', x_j) \\ &- \gamma^{(k+1)}(t, x_1, \dots, x_k, x_j'; x_1', \dots, x_j', \dots, x_k', x_j').\end{aligned}$$

Factorized solutions of the GP hierarchy

It is easy to see that

$$\gamma^{(k)} = |\,\phi\,\rangle\langle\,\phi\,|^{\otimes k} := \prod_{j=1}^{k} \phi(t, x_j)\,\overline{\phi(t, x_j')}$$

is a solution of the GP if ϕ satisfies the cubic NLS

$$i\partial_t \phi + \Delta_x \phi - b_0\,|\phi|^2\,\phi = 0$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Step 2: Uniqueness of solutions to the GP hierarchy

- One considers the r-fold iterate of the Duhamel formula for $\gamma^{(k)}$, with initial data $\gamma_0^{(k)} = 0$, for some arbitrary $r \in \mathbb{N}$,

$$\gamma^{(k)}(t) = (i\lambda)^r \int_{t \geq t_1 \geq \cdots \geq t_r} dt_1 \cdots dt_r U^{(k)}(t - t_1) B_{k+1} U^{(k+1)}(t_1 - t_2) \cdots \\ \cdots U^{(k+r-1)}(t_{r-1} - t_r) B_{k+r} \gamma^{(k+r)}(t_r) \tag{2.5}$$

- **A key difficulty** stems from the fact that the interaction operator $B_{\ell+1}$ is the sum of $O(\ell)$ terms, therefore (2.5) contains $O(\frac{(k+r-1)!}{(k-1)!}) = O(r!)$ terms.

- The proof of uniqueness is accomplished by using highly sophisticated **Feynman graphs**. Solutions of the GP hierarchy are studied in "Trace Sobolev" spaces equipped with norms

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|S^{(k,\alpha)} \gamma^{(k)}|),$$

where $S^{(k,\alpha)} := \prod_{j=1}^k \langle \nabla_{x_j} \rangle^\alpha \langle \nabla_{x_j'} \rangle^\alpha$.

Uniqueness of GP following Klainerman-Machedon

- *Klainerman and Machedon* (2008) introduced an alternative method for proving uniqueness in the space:

$$\|\gamma^{(k)}\|_{H_k^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}.$$

- The method is based on:
  - the **“board game argument”** and
  - the use of certain **space-time estimates**
- The results is conditional, assuming

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L_t^1 \dot{H}_k^1} < C^k. \tag{2.6}$$

- Subsequently:
  - *Kirkpatrick, Schlein and Staffilani* (2011) were the first to use the KM formulation to derive the cubic NLS in $d = 2$ via proving that the limit of the BBGKY satisfies (2.6).
  - *Chen-P* (2011) generalized this to derive the quintic GP in $d = 1, 2$.
  - *Xie* (2013) generalized it to derive a general power-type NLS.
  - A derivation of the cubic NLS in $d = 3$ in the KM spaces was settled by *Chen-P*, *X. Chen*, *X. Chen-Holmer*, *T. Chen-Taliaferro*.

Analysis of the GP inspired by the NLS

Since the GP

- arises in a derivation of NLS from quantum many-body systems

it is natural to ask:

1. Whether the GP retains some features of a dispersive PDE?
2. Whether methods of nonlinear dispersive PDE can be “lifted” to the GP and the QFT levels?

From NLS “to” the GP then

From NLS “to” the GP: = Analysis of the GP inspired by the NLS

Dispersive tools at the level of the GP

1. Tools at the level of the GP, that are inspired by the NLS techniques, are instrumental in understanding:
   - Well-posedness for the GP hierarchy
   - Well-posedness for quantum many body systems
   - Going from bosons to NLS in Klainerman-Machedon spaces

   Results of: *Chen-P, Chen-P-Tzirakis, Chen-Taliaferro, Gressman-Sohinger-Staffilani, Sohinger, X. Chen, X. Chen-Holmer*.

2. But there were still few questions that resisted the efforts to apply newly built tools at the level of the GP, e.g.
   - Long time behavior of the GP hierarchy
   - Uniqueness of the cubic GP on $\mathbb{T}^3$
   - Uniqueness of the quintic GP on $\mathbb{R}^3$

From NLS “to” the GP now

The key tool: quantum de Finetti theorem.

Quantum de Finetti as a bridge between the NLS and the GP

What is quantum De Finetti?

Strong quantum de Finetti theorem

Due to: *Hudson-Moody (1976/77), Stormer (1969), Lewin-Nam-Rougerie (2013)*

Theorem

(Strong Quantum de Finetti theorem) Let $\mathcal{H}$ be any separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{sym}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Let Γ denote a collection of admissible bosonic density matrices on $\mathcal{H}$, i.e.,

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on $\mathcal{H}^k$, and $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$. Then, there exists a unique Borel probability measure μ, supported on the unit sphere $S \subset \mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N} . \tag{3.1}$$

Applications of quantum de Finetti theorem

We shall focus on two specific applications of quantum de Finetti:

1. Uniqueness of solutions to the GP via quantum de Finetti
2. Infinitely many conservation laws for the cubic GP hierarchy

Uniqueness of solutions to the GP via quantum de Finetti theorems

- Until recently, the only available proof of unconditional uniqueness of solutions in[3] $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the cubic GP hierarchy in $\mathbb{R}^3$ was given in the works of Erdös, Schlein, and Yau, who developed an approach based on use of Feynman graphs.
- Together with T. Chen, C. Hainzl and R. Seiringer, we obtained a new proof based on quantum de Finetti theorem.

[3] The $\mathfrak{H}^1$ denotes the trace class Sobolev space defined for the entire sequence $(\gamma^{(k)})_{k\in\mathbb{N}}$:

$$\mathfrak{H}^1 := \left\{ (\gamma^{(k)})_{k\in\mathbb{N}} \,\middle|\, \mathrm{Tr}(|S^{(k,1)}\gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \right\}.$$

Statement of the result

Theorem (Chen-Hainzl-P-Seiringer)

Let $(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ be a mild solution in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ to the (de)focusing cubic GP hierarchy in $\mathbb{R}^3$ with initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$, which is either admissible, or obtained at each t from a weak- limit.*
Then, $(\gamma^{(k)})_{k\in\mathbb{N}}$ is the unique solution for the given initial data.
Moreover, assume that the initial data $(\gamma^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1$ satisfy

$$\gamma^{(k)}(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,, \tag{3.2}$$

where μ is a Borel probability measure supported either on the unit sphere or on the unit ball in $L^2(\mathbb{R}^3)$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one. Then,

$$\gamma^{(k)}(t) = \int d\mu(\phi)(|S_t(\phi)\rangle\langle S_t(\phi)|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}\,, \tag{3.3}$$

where $S_t : \phi \mapsto \phi_t$ is the flow map of the cubic (de)focusing NLS.

Setup of the proof

Assume that we have two positive semidefinite solutions $(\gamma_j^{(k)}(t))_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^1$ satisfying the same initial data,

$$(\gamma_1^{(k)}(0))_{k\in\mathbb{N}} = (\gamma_2^{(k)}(0))_{k\in\mathbb{N}} \in \mathfrak{H}^1.$$

Then,

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t) \quad , \quad k \in \mathbb{N}, \tag{3.4}$$

is a solution to the GP hierarchy with initial data $\gamma^{(k)}(0) = 0 \ \forall k \in \mathbb{N}$, and it suffices to prove that

$$\gamma^{(k)}(t) = 0$$

for all $k \in \mathbb{N}$, and for all $t \in [0, T)$.

Via de Finetti

From de Finetti theorems, we have

$$\gamma_j^{(k)}(t) = \int d\mu_t^{(j)}(\phi) \big(|\phi\rangle\langle\phi|\big)^{\otimes k},$$

for $j = 1, 2$ and therefore

$$\gamma^{(k)}(t) = \int d\widetilde{\mu}_t(\phi) \big(|\phi\rangle\langle\phi|\big)^{\otimes k},$$

where $\widetilde{\mu}_t := \mu_t^{(1)} - \mu_t^{(2)}$ is the difference of two probability measures on the unit ball in $L^2(\mathbb{R}^3)$.

Representation of solution using KM and de Finetti

KM implies that we can represent $\gamma^{(k)}(t)$ in upper echelon form:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1 \cdots dt_r U^{(k)}(t-t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1-t_2) \cdots \\ \cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{\sigma(k+r),k+r} \gamma^{(k+r)}(t_r)$$

Now using the quantum de Finetti theorem, we obtain:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{N}_{k,r}} \int_{D(\sigma,t)} dt_1, \ldots, dt_r \int d\widetilde{\mu}_{t_r}(\phi)\, J^k(\sigma; t, t_1, \ldots, t_r),$$

where

$$J^k(\sigma; t, t_1, \ldots, t_r; \underline{x}_k; \underline{x}'_k) = \Big(U^{(k)}(t-t_1) B_{\sigma(k+1),k+1} U^{(k+1)}(t_1-t_2) \cdots \\ \cdots U^{(k+r-1)}(t_{r-1}-t_r) B_{\sigma(k+r),k+r} \big(|\phi\rangle\langle\phi|\big)^{\otimes(k+r)} \Big)(\underline{x}_k; \underline{x}'_k).$$

Roadmap of the proof

1. recognize that a certain product structure gets preserved from right to left along iterated Duhamel formulas
2. get an estimate on integrals in upper echelon form via recursively performing Strichartz estimates (at the level of the Schrödinger equation) from left to right

Open questions for the GP revisited

Questions that resisted the pre-de-Finetti efforts became accessible now:

- Long time behavior of the GP hierarchy (Chen-Hainzl-P.-Seiringer)
- Uniqueness of the cubic GP on $\mathbb{T}^3$ (Sohinger)
- Uniqueness of the quintic GP on $\mathbb{R}^3$ (Hong-Taliaferro-Xie)

Infinitely many conservation laws for the cubic GP hierarchy

The cubic NLS in 1D

$$i\phi_t + \partial_{xx}\phi = 2\kappa|\phi|^2\phi \tag{3.5}$$

has been extensively studied

- as a dispersive equation and
- as an example of an integrable equation
  - Zakharov-Shabat (1972)
  - Ablowitz-Kaup-Newell-Segur (1974)
  - Zakharov-Manakov (1976)
  - Its (1982)
  - ...

There are many consequences of integrability for an equation, but perhaps the most well-known is the existence of infinitely many conserved quantities.

Infinitely many conserved quantities for the cubic GP hierarchy

Inspired by the construction of infinitely many conserved quantities for the cubic NLS in 1D, with D. Mendelson, A. Nahmod and G. Staffilani we **identify infinitely many conserved quantities for smooth solutions for the GP.**

- This may provide a first step towards understanding a possible integrable structure for the GP, and perhaps eventually understanding a physical derivation of the integrable structure for the cubic NLS.
- We also view this work from the perspective of the program of passing results from nonlinear PDE, such as the NLS to the infinite particle system given via the GP hierarchy.

Infinitely many conservation laws for the cubic NLS

Here we follow Faddeev-Takhtajan's presentation (based on the inverse scattering of Zakharov-Shabat 1972).

Definition

Let $\phi \in C_0^\infty$ be a solution to the cubic NLS (3.5). *For a non-negative integer n we define the functions w_{n+1} by:*

$$w_1(x) = \phi(x) \tag{3.6}$$

$$w_{n+1}(x) = -i\partial_x w_n(x) + \kappa\overline{\phi}(x) \sum_{k=1}^{n-1} w_k(x) w_{n-k}(x). \tag{3.7}$$

Let $\phi \in C_0^\infty$ be a solution to the cubic NLS. Then for each $n \in \mathbb{N}$, the quantity

$$I_n(x) := \int w_n(x)\overline{\phi}(x)\, dx \tag{3.8}$$

is conserved in time.

The operators $\{\mathbf{W}_n^j\}_n$ for the GP

In an analogy to functions $\{w_n\}_{n\in\mathbb{N}}$, we introduce the operators $\{\mathbf{W}_n^j\}_n$.

We record explicitly the first few operators $\mathbf{W}_n^j$ to illustrate their form:

$$
\begin{aligned}
\mathbf{W}_1^j &= I_j \\
\mathbf{W}_2^j &= -i\partial_{x_j}\mathrm{Tr}_{j+1} \\
\mathbf{W}_3^j &= -\partial_{x_j}^2\mathrm{Tr}_{j+1j+2} + \kappa B_{j,j+1}\mathrm{Tr}_{j+2} \\
\mathbf{W}_4^j &= i\partial_{x_j}^3\mathrm{Tr}_{j+1,j+2,j+3} - i\kappa(B_{j,j+1}\partial_{x_{j+1}}\mathrm{Tr}_{j+2} + B_{j,j+2}\partial_{x_j}\mathrm{Tr}_{j+1})\mathrm{Tr}_{j+3} \\
\mathbf{W}_5^j &= \partial_{x_j}^4\mathrm{Tr}_{j+2,j+3,j+4,j+5} + \kappa\Big[B_{j;j+1}(-\partial_{x_{j+1}}^2\mathrm{Tr}_{j+2,j+3} + \kappa B_{j+1,j+2}\mathrm{Tr}_{j+3}) \\
&\qquad - B_{j;j+2}\partial_{x_j}\mathrm{Tr}_{j+1}\partial_{x_{j+2}}\mathrm{Tr}_{j+3} + B_{j;j+3}(-\partial_{x_j}^2\mathrm{Tr}_{j+1,j+2} + \kappa B_{j,j+1}\mathrm{Tr}_{j+2})\Big]\mathrm{Tr}_{j+4}.
\end{aligned}
$$

We note that $\mathbf{W}_3^j$ is the operator associated to the conserved energy identified in Chen-P-Tzirakis, and used to construct higher order energies in Chen-P.

Main result - "get to the point" version

Theorem (Mendelson-Nahmod-P.-Staffilani)

Let $(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ be a sufficiently regular solutions to the GP hierarchy. Then for each $n \in \mathbb{N}$, $1 \leq j \leq n$ such that $k \geq j+n-1$ we have that the quantity

$$\operatorname{Tr} \boldsymbol{W}_n^j \gamma^{(k)}(t)$$

is conserved in time, that is, for any $t \in \mathbb{R}$,

$$\operatorname{Tr} \boldsymbol{W}_n^j \gamma^{(k)}(t) = \operatorname{Tr} \boldsymbol{W}_n^j \gamma^{(k)}(0).$$

How do we introduce the operators $\{\mathbf{W}^j_n\}_n$?

1. First via a recursive definition for factorized solutions of the GP.
2. Subsequently we extend the definition to general solutions to the GP via
   - 1D version of de Finetti type uniqueness theorem for the GP, due to Hong-Taliaferro-Xie
   - functional analytical tools related to Bochner integral.

Zoom into the result for factorized solutions - Definition of $\{\mathbf{W}^j_n\}_n$ operators

For a non-negative integer n and any integer $1 \leq j \leq n+1$ and $\alpha > 0$ we define the operators

$$\mathbf{W}^j_{n+1} : \mathfrak{h}^{n/2+\alpha}_{n+1} \to \mathfrak{h}^{\alpha}_1$$

by:

$$\mathbf{W}^1_1\left(\phi(x_1)\overline{\phi}(x_1')\right) = \phi(x_1)\overline{\phi}(x_1')$$

$$\mathbf{W}^j_{n+1}\left(\prod_{l=j}^{j+n}\phi(x_l)\overline{\phi}(x_l')\right) = -i\partial_{x_j}\mathbf{W}^j_n \mathrm{Tr}_{x_{j+n}}\left(\prod_{l=1}^{j+n}\phi(x_l)\overline{\phi}(x_l')\right)$$

$$+\,\kappa\sum_{k=1}^{n-1} B_{j,j+k}\mathbf{W}^j_k \otimes \mathbf{W}^{j+k}_{n-k}\mathrm{Tr}_{x_{n+j}}\left(\prod_{l=j}^{j+k-1}\phi(x_l)\overline{\phi}(x_l')\prod_{l=j+k}^{j+n-1}\phi(x_l)\overline{\phi}(x_l')\right)\phi(x_{j+n})\overline{\phi}(x_{j+n}'),$$

for any $\phi \in H^{n/2+\alpha}$ and such that $\|\phi\|_{L^2} = 1$.

Zoom into the result for factorized solutions - Conserved quantitates

The crucial property of the operators $\{\mathbf{W}_n^j\}_n$:

$$\mathbf{W}_n^j\left(\prod_{l=j}^{j+n-1}\phi(t,x_l)\overline{\phi}(t,x_l')\right) = w_n(x_j)\overline{\phi}(t,x_j'), \tag{3.9}$$

for every $n = 0, 1, 2, ..., 1 \leq j \leq n+1$ and $\phi(t,x)$ a solution to the cubic defocusing NLS in a certain sufficiently regular Sobolev space.

In particular,

$$\mathbf{W}_n^1\left(\prod_{l=1}^{n}\phi(t,x_l)\overline{\phi}(t,x_l')\right) = w_n(x_1)\overline{\phi}(t,x_1'). \tag{3.10}$$

Now (3.10) with the NLS result (3.8) implies that for each $n \in \mathbb{N}$, the quantity

$$\mathrm{Tr}\mathbf{W}_n^1\left(\prod_{l=1}^{n}\phi(t,x_l)\overline{\phi}(t,x_l')\right) = \int w_n(x)\overline{\phi}(t,x)\,dx = I_n(x) \tag{3.11}$$

is conserved in time.

Directions that we plan to consider:

- Understand a possible integrable structure for the GP.
- Eventually understand a physical derivation of the integrable structure for the cubic NLS.

Back and forth from many body systems to nonlinear equations

Other examples:

- “From Newton to Boltzmann: hard spheres and short-range potentials”
  Gallagher - Saint-Raymond -Texier, 2012

- “Kac’s Program in Kinetic Theory”
  Mischler - Mouhot, 2011

Thank you!