**Nejla Nouaili**

*CEREMADE, Université Paris Dauphine.*


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# Interests

Formation of singularities in finite time in partial differential equation of parabolic type:

- Nonlinear heat equation.
- Complex Ginzburg equation.

# Actual interest: The complex Ginzburg Landau (CGL) equation

We consider the following equation

$$\begin{array}{rcl} \partial_t u & = & (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \gamma u \\ u(x,0) & = & u_0(x) \text{ for } x \in \mathbb{R}, \end{array} \qquad (CGL)$$

where

- $p > 1$, $\beta$, $\delta$ and $\gamma$ are reals.
- $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{C}$.
- $u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C})$.

# Mathematical relevance for CGL

- Mathematical relevance: Classical tools break down:
  - Maximum principle;
  - Variational formulation;
  - Energy methods.

# History of blow-up in CGL equation

- $p = 3$, Formal approach Existence of blow-up solutions and blow-up behavior was obtained by Hocking and Stewartson (1972), Popp, Stiller, Kuznetsov and Kramer (1998), under some condition on $\beta$ and $\delta$:
  - Existence of blow-up solutions;
  - Determination of the blow-up Behavior.
- Rigorous approach for $p > 1$: Construction, profile and stability, under some conditions on $\beta$ and $\delta$,
  - when $\beta = 0$, see Zaag (1998);
  - when $\beta \neq 0$, see Masmoudi and Zaag (2008).
- Case $\beta = \delta$: This is variational. Results by Cazenave, Dickstein and Weissler 2012.

# Case $\beta = \delta = 0$, the heat equation

- The generic profile is given by

$$(T-t)^{\frac{1}{p-1}} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f_0(z),$$

where $f_0(z) = (p - 1 + b_0|z|^2)^{-\frac{1}{p-1}}$ and $b_0 = \frac{(p-1)^2}{4p}$
See Galaktionov-Posashkov (1985), Berger-Kohn (1988), Herrero-Velzquez (1993). The constructive existence proof by Bricmont-Kupiainen (1994), Merle-Zaag. (1997) .

# Case $\beta \neq 0$ and $\delta \neq 0$

If

$$p - \delta^2 - \beta\delta(p+1) > 0, \qquad (Subcritical)$$

then, Masmoudi and Zaag (2008):

- Constructed a solution such that

$$(T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(z\sqrt{(T-t)|\log(T-t)|}, t) \sim f(z),$$

where $f(z) = \kappa^{-i\delta}(p-1+b|z|^2)^{-\frac{1+i\delta}{p-1}}$, $\kappa = (p-1)^{-\frac{1}{p-1}}$

$$b = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(p+1))} \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2);$$

- proved the stability with respect to initial data.

**Question: What happens in the critical case?**

Theorem (Nouaili and Zaag, Existence of a blow-up solution with determination of its profile)

*If*

$$p = \delta^2,$$

*then, there exists a solution* $u(x,t)$*, s.t.*

- *Blow-up profile*

$$(T-t)^{\frac{1+i\delta}{p-1}}|\log(T-t)|^{-i\mu}u(z\sqrt{(T-t)}|\log(T-t)|^{\frac{1}{4}} \sim f_c(z) \text{ as } t \to T,$$

*where*

$$f_c(z) = (p-1+b_c|z|^2)^{-\frac{1+i\delta}{p-1}},$$

$$b_c = \frac{(p-1)^2}{8\sqrt{p(p+1)}} \text{ and } \mu = \frac{8\delta b^2}{(p-1)^4}(1+p).$$

# Idea of the proof

We follow the the constructive existence proof used by Bricomont-Kupiainen (1994), Merle-Zaag (1997) for standard semilinear heat equation and Masmoudi and Zaag (2008) for the CGL equation in the subcritical case.

The method is base on:

- The reduction of the problem to a finite-dimensional one ($N + 1$ parameters);
- The solution of the finite-dimensional problem thanks to the degree theory.

# Stability of the constructed solution

Thanks to the interpretation of the $(N+1)$ parameters of the finite-dimensional problem in terms of the blow-up time in $(\mathbb{R})$ and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

**Theorem (Nouaili and Zaag: Stability)**

*The constructed solution is stable with respect to perturbation in initial data.*
*Consider initial data $\hat{u}_0$ of the solution of (CGL) with blow-up time $\hat{T}$, blow-up point $\hat{a}$ and profile $f_c$ centred at $(\hat{T}, \hat{a})$.*
*Then, $\exists \mathcal{V}$ neighborhood of $\hat{u}_0$ s.t. $\forall u_0 \in \mathcal{V}$, $u(x,t)$ the solution of (CGL) blows up at time $T$, at a point $a$, with the profile $f_c$ centred at $(T, a)$.*

Work in progress:

- Generalization of our result to the case $\beta \neq 0$, with H.Zaag and K.Duong.

Future direction:

- Construction of new blow-up behaviors.
- To understand how the collapse of a solution may be suppressed for suitable parameters $\beta$, $\delta$.