A Rigidity result for the Camassa-Holm equation and applications

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Presentation of the equation

The Camassa-Holm equation reads

$$(CH) \quad \begin{cases} u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{3x} \ , \ (t,x) \in \mathbb{R}^2 \ , \\ u(0,x) = u_0(x) \ , \end{cases}$$

where $u(t,x)$ is real-valued.

It has been derived in 93' by Camassa and Holm, starting from the Green-Naghdi equations and making an asymptotic expansion that keeps the hamiltonian structure.

Rigorous derivation from the full water waves problem obtained by Constantin and Lannes 2009'.

Presentation of the equation

A lot of properties :

- An infinite number of conservation laws.

$$M(u) = \int_{\mathbb{R}} (u - u_{xx})\, dx\,, \quad E(u) = \int_{\mathbb{R}} u^2 + u_x^2$$

$$F(u) = \int_{\mathbb{R}} u^3 + uu_x^2$$

  The equation may be rewritten in Hamiltonian form :

$$\partial_t E'(u) = -\partial_x F'(u) \quad .$$

- Non smooth solitary waves

$$u(t,x) = ce^{-|x-ct|} = \varphi_c(x-ct), \quad c \in \mathbb{R}^*$$

  where $\varphi_c = ce^{-|x]}$ is the unique H^1-weak solutions to

$$-c\varphi_c + c\varphi_c'' + \frac{3}{2}\varphi_c^2 = \varphi_c\varphi_c'' + \frac{1}{2}(\varphi_c')^2$$

Presentation of the equation

To give a sense to the peakon-solutions one rewrites (CH) as

$$u_t + uu_x + (1-\partial_x^2)^{-1}\partial_x(u^2 + u_x^2/2) = 0$$

It is also worth noticing that the momentum density $y = u - u_{xx}$ satisfies the transport equation

$$y_t + uy_x + 2u_x y = 0$$

Local well-posedness results

- Locally well-posed in $H^s(\mathbb{R})$ for $s > 3/2$.
- There exist solutions that blow up in finite time by wave breaking

$$\liminf_{t \nearrow T^*} u_x(t,x) = -\infty\ .$$

Prob : $e^{-|x|} \notin H^{3/2}(\mathbb{R})$!

Well-posedness results

Theorem (Constantin-M 00')

Let $u_0 \in H^1(\mathbb{R})$ with $y_0 = u_0 - u_{0,xx} \in \mathcal{M}_+(\mathbb{R})$ then $\exists!$ solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ such that $y = u - u_{xx} \in L^\infty(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$. Moreover, M, E and F are conserved along the flow.

We set $Y_+ := \{u \in H^1(\mathbb{R}), u - u_{xx} \in \mathcal{M}_+(\mathbb{R})\}$.
Note that $e^{-|x|} \in Y_+$ since $(1 - \partial_x^2)e^{-|x|} = 2\delta_0$

Theorem (Constantin-Strauss 00')

Let $u \in C([0, T]; H^1(\mathbb{R}))$ such that

$$\|u_0 - ce^{-|x|}\|_{H^1} < \varepsilon^4 \le \varepsilon_0^4$$

then

$$\sup_{t\in[0,T]} \|u(t) - ce^{-|x-\xi(t)|}\|_{H^1} < O(\varepsilon)$$

where $\xi(t)$ is any point where $u(t)$ reaches it maximum $(c > 0)$.

Presentation of the results

Definition

We say that a solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$ of (C-H) is Y-almost localized if there exist $c > 0$ and a C^1-function $x(\cdot)$, with $x_t \geq c > 0$, for which for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for all $t \in \mathbb{R}$ and all $\Phi \in C(\mathbb{R})$ with $0 \leq \Phi \leq 1$ and $\text{supp}\Phi \subset [-R_\varepsilon, R_\varepsilon]^c$.

$$\int_{\mathbb{R}} (u^2(t) + u_x^2(t))\Phi(\cdot - x(t))\, dx + \Big\langle \Phi(\cdot - x(t)), u(t) - u_{xx}(t) \Big\rangle \leq \varepsilon \,. \tag{1}$$

Presentation of the results

Theorem (rigidity property)

Let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$*, with* $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$*, be a* Y*-almost localized solution of (C-H) that is not identically vanishing. Then there exists* $c^* > 0$ *and* $x_0 \in \mathbb{R}$ *such that*

$$u(t) = c^* \varphi(\cdot - x_0 - c^* t), \quad \forall t \in \mathbb{R}\,.$$

Presentation of the results

Theorem (asymptotic stability)

Let $c > 0$ be fixed. There exists an universal constant $0 < \eta \ll 1$ such that for any $0 < \theta < c$ and any $u_0 \in Y_+$ satisfying

$$\|u_0 - \varphi_c\|_{H^1} \leq \eta \Big(\frac{\theta}{c}\Big)^8 , \tag{2}$$

there exists $c^* > 0$ with $|c - c^*| \ll c$ and a C^1-function $x : \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to\infty} \dot{x} = c^*$ such that

$$u(t, \cdot + x(t)) \underset{t\to+\infty}{\rightharpoonup} \varphi_{c^*} \text{ in } H^1(\mathbb{R}) , \tag{3}$$

where $u \in C(\mathbb{R}; H^1)$ is the solution emanating from u_0. Moreover,

$$\lim_{t\to+\infty} \|u(t) - \varphi_{c^*}(\cdot - x(t))\|_{H^1(]\theta t,+\infty[)} = 0 . \tag{4}$$

Presentation of the results

Using that (C-H) is invariant by the change of unknown $u(t,x) \mapsto -u(t,-x)$, we obtain as well the asymptotic stability of the antipeakon profile $c\varphi$ with $c < 0$ in the class of H^1-function with a momentum density that belongs to $\mathcal{M}_-(\mathbb{R})$.

This theorem implies the growth of the high Sobolev norms for some smooth solutions of the Camassa-Holm equation. Indeed, it follows from this theorem that any solution of the Camassa-Holm equation emanating from an initial datum $u_0 \in Y_+ \cap H^s(\mathbb{R})$, $s \geq 3/2$, satisfying (2), has a $H^s(\mathbb{R})$-norm that tends to $+\infty$ as t tends to infinity.

Presentation of the results

- The proof of the rigidity result uses the finite speed propagation of the momentum density y.

- The proof of the asymptotic stability follows the framework developed by Martel and Merle.

Proof of the asymptotic stability

Let $u_0 \in Y_+$ such that

$$\|u_0 - c\varphi\|_{H^1} < \varepsilon^8$$

$\exists!$ C^1-function $x(\cdot)$ with $|\dot{x}(t) - c| \ll c$ and

$$\|u(t,\cdot) - c\varphi(\cdot - x(t))\|_{H^1} = O(\varepsilon)$$

such that

$$\int_{\mathbb{R}} \varphi'(\cdot - x(t))u(t,\cdot) = 0, \quad \forall t \in \mathbb{R} .$$

Let $\{t_n\} \nearrow +\infty$. By Ascoli theorem

$$x(t_n + \cdot) - x(t_n) \longrightarrow \tilde{x} \text{ in } \; C(-T,T]$$

and by local compactness ($Y \hookrightarrow H^{\frac{3}{2}-}(\mathbb{R})$)

$$u(t_n, \cdot + x(t_n)) \longrightarrow \tilde{u}_0 \;\text{ in }\; H^1_{loc}(\mathbb{R})$$

Proof of the asymptotic stability

Denoting by $\tilde{u}$ the solution of (C-H) emanating from $\tilde{u}_0$ this yields

$$u(t_n + t, \cdot + x(t_n + t)) \longrightarrow \tilde{u}(t, \cdot + \tilde{x}(t)) \text{ in } H^1_{loc}(\mathbb{R}), \quad \forall t \in \mathbb{R},$$

where we used a continuous dependence result for (C-H) with respect to the weak H^1-topology.
This enables to prove that $\tilde{u}$ is an Y-almost localized solution and thus

$$\tilde{u}_0 = c_0 \varphi(\cdot - x_0)$$

It remains to prove that c_0 and x_0 does not depend on $\{t_n\}$.

First the orthogonality condition forces $x_0 = 0$.

Proof of the asymptotic stability

Now, since there is local strong convergence in $L^\infty(\mathbb{R})$ we must hat

$$\max u(t_n,\cdot) \to c_0$$

We set $\lambda(t) = \max_{\mathbb{R}} u(t)$ so that

$$u(t_n, \cdot + x(t_n)) - \lambda(t_n)\varphi \underset{n\to+\infty}{\rightharpoonup} 0 \ \text{ in } H^1(\mathbb{R})$$

Since this is true for any $\{t_n\} \nearrow \infty$ we get that

$$u(t, \cdot + x(t)) - \lambda(t)\varphi \underset{t\to+\infty}{\rightharpoonup} 0 \ \text{ in } H^1(\mathbb{R})$$

It remains to prove that $\lambda(t) \to c^*$. This uses an almost monotonicity result or the part of E that travels at the right or the left of an almost localized solution

Proof of the rigidity result

Step 1: Uniform exponential decay of Y localized solutions.

This is a consequence of almost monotonicity results for the parts of E and M that travel at the right or the left of an almost localized solution.

Step 2: Proof of the compact support of y at the right side.

Let $q(\cdot,\cdot)$ be the flow associated with u

$$\begin{cases} q_t(t,x) &= u(t,q(t,x)) \quad , \ (t,x) \in \mathbb{R}^2 \\ q(0,x) &= x \quad , \ x \in \mathbb{R} \end{cases} .$$

$$y_t + uy_x = -2u_x y \Rightarrow \partial_t \Big(y(t,q(t,\cdot)) e^{2\int_0^t u_x(s,q(s,\cdot))\,ds} \Big) = 0$$

On the other hand $\partial_x q(0,x) = 1$ and

$$\partial_t q_x(t,x) = q_x(t,x) u_x(t,q(t,x))$$

ensure that

$$q_x(t,x) = \exp\Big(\int_0^t u_x(s,q(s,x))\,ds \Big)$$

Proof of the rigidity result

This yields

$$\forall t \in \mathbb{R}, \quad y(t, q(t,\cdot))q_x^2(t,\cdot) = y(0,\cdot)\ .$$

By the Y localization of u there exists $R_0 > 0$ such that

$$\forall t \in \mathbb{R}, \forall |x| > R_0, \quad u(t, x(t) + R_0) < \frac{c}{10}$$

In particular $\frac{d}{dt}_{/t=0} q(t, x(0) + R_0) = u(0, x(0) + R_0) < \frac{c}{10}$ and by continuity

$$\forall t \leq 0, \quad q(t, x(t) + R_0) - x(t) \geq R_0 + \frac{c}{2}|t|$$

Combining this with $|u_x| \leq u$ and the exponential decay we get

$$\forall t \leq 0, \forall x \geq 0, \quad |u_x(t, q(t, x(0) + R + x)| \leq Ce^{-\beta(R+|t|)}$$

Proof of the rigidity result

This ensures that for $\forall t \leq 0$, $\forall x \geq 0$,

$$\frac{1}{C_0} \leq q_x(t, x(0) + R_0 + x) \leq C_0$$

Assume that $y(0)$ is not compactly supported at the right. Then there exists $R > R_0$ such that

$$\int_{x(0)+R_0}^{x(0)+R} y(0,x)dx = \varepsilon_0 > 0$$

$$\Rightarrow \int_{x(0)+R_0}^{x(0)+R} y(t,q(t,x))q_x(,t,x)^2\,dx = \varepsilon_0$$

$$\Rightarrow \int_{x(0)+R}^{x(0)+R} y(t,q(t,x))q_x(,t,x)\,dx \geq \frac{\varepsilon_0}{C_0}$$

and performing the change of variables $z = q(t,x)$

$$\int_{q(t,x(0)+R_0)}^{q(t,x(0)+R)} y(t,z)dz \geq \frac{\varepsilon_0}{C_0} \Rightarrow \int_{x(t)+R_0+c|t|/2}^{+\infty} y(t,z)dz \geq \frac{\varepsilon_0}{C_0}$$

that contradicts the Y-localization of u as $t \to -\infty$.

Proof of the rigidity result

Therefore $\operatorname{supp} y(t) \subset [-\infty, x(t)+R_0]$ for all $t \in \mathbb{R}$. Now it will be useful to notice that

$$u(t, x(t)+r_0) = -u_x(t, x(t)+R_0) \geq \frac{e^{-2r_0}}{4\sqrt{R_0}}\sqrt{E(u)} = \alpha_0\,.$$

Indeed, by the Y-localization of u, the conservation of $E(u)$ and the choice of R_0

$$\|u(t,\cdot - x(t))\|_{H^1(]-R_0,R_0[)} \geq \frac{1}{2}\sqrt{E(u)}\,.$$

But $y = u - u_{xx} \geq 0$ ensures that $-u \leq u_x \leq u$ on $\mathbb{R}$. This forces

$$\max_{[-r_0,r_0]} u^2(t,\cdot - x(t)) \geq \frac{1}{2r_0}\|u(t,\cdot - x(t))\|^2_{L^2]-r_0,r_0[)} \geq \frac{1}{8r_0}E(u)$$

But since $u_x \geq -u$ on $\mathbb{R}^2$, for any $(t, x_0) \in \mathbb{R}^2$ it holds

$$u(t,x) \leq u(t,x_0)e^{-x+x_0}, \quad \forall x \leq x_0\,.$$

Proof of the rigidity result

Applying this estimate with $x_0 = x(t) + R_0$ we obtain that

$$u(t, x(t) + R_0) \geq \max_{[-R_0, R_0]} u(t, \cdot - x(t))e^{-2R_0}$$

which yields the desired result.
Now we set

$$x_+(t) = \inf\{x \in \mathbb{R}, \operatorname{supp} y(t) \subset]-\infty, x(t) + x]\}$$

and

$$q^*(t) = q(t, x(0) + x_+(0)) = x(t) + x_+(t)$$

Proof of the rigidity result

Step 3: Study of the jump of $u_x(t,\cdot)$ at $q^*(t)$.

We set

$$a(t) = u_x(t, q^*(t)-) - u_x(t, q^*(t)+), \quad \forall t \in \mathbb{R}\,. \tag{5}$$

Then $a(\cdot)$ is a bounded non decreasing derivable function on $\mathbb{R}$ with values in $[\frac{\alpha_0}{8}, 2\sqrt{E(u)}]$ such that

$$a'(t) = \frac{1}{2}(u^2 - u_x^2)(t, q^*(t)-), \; \forall t \in \mathbb{R}. \tag{6}$$

First we prove that $u_x(t)$ has got a jump at $q^*(t)$. We proceed by contradiction assuming that there exists $x_1 < q^*(0)$ such that $\|y(0)\|_{\mathcal{M}(]x_1,+\infty[)} < \alpha/8$.

Proof of the rigidity result

On $]x_1, q^*(0)[$ it holds

$$\begin{aligned} u_x(0,x) &\leq -\alpha_0 - \int_x^{q^*(0)} u_{xx} \\ &\leq -\alpha_0 - \int_x^{q^*(0)} u + \int_x^{q^*(0)} y \\ &\leq -3\alpha_0/4 \end{aligned}$$

Proof of the rigidity result

This ensures that $q_x(0,x) \geq 1$ on $]x_1, q^*(0)[$.
We can extend this for any $t < 0$ on $]q(t,x_1), q^*(t)[$ since

$$
\begin{aligned}
u_x(t,x) &\leq -\alpha_0 + \int_x^{q^*(t)} y \\
&\leq -\alpha_0 + \int_{q^{-1}(t,x)}^{q^*(0)} y(t,q(t))q_x(t,x)\,dx \\
&\leq -\alpha_0 + \int_{x_1}^{q^*(0)} y(t,q(t))q_x^2(t,x)\,dx \\
&\leq -\alpha_0 + \int_{x_1}^{q^*(0)} y(0,x)\,dx \\
&\leq -3\alpha_0/4
\end{aligned}
$$

Proof of the rigidity result

This forces $q^*(t) - q(t, x_1) \to +\infty$ as $t \to -\infty$ and $u(t, q(t, x_1)) \geq u(t, q^*(t)) \geq \alpha_0$ that contradicts the almost localization of u .